2D elastic equations in a VTI medium

- We recall the general form of the 3D elastodynamics equations

$$\begin{aligned} \rho \frac{\partial v}{\partial t} &= D\sigma + F \\ \frac{\partial \sigma}{\partial t} &= CD^T v + \frac{\partial \sigma^0}{\partial t} \end{aligned} \tag{43}$$

with

$$v = [v_x \;\; v_y \;\; v_z]^T, \;\; \sigma = [\sigma_{xx} \;\; \sigma_{yy} \;\; \sigma_{zz} \;\; \sigma_{yz} \;\; \sigma_{xz} \;\; \sigma_{xy}]^T \tag{44}$$

$$C = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{pmatrix}, \;\; D = \begin{pmatrix} \partial x_1 & 0 & 0 & 0 & \partial x_2 & \partial x_3 \\ 0 & \partial x_2 & 0 & \partial x_3 & 0 & \partial x_1 \\ 0 & 0 & \partial x_3 & \partial x_2 & \partial x_1 & 0 \end{pmatrix}. \tag{45}$$

2D elastic equations in a VTI medium

- For the 2D P-SV system we have (using invariance along y axis)

$$\begin{aligned} \rho \frac{\partial v}{\partial t} &= D\sigma + F \\ \frac{\partial \sigma}{\partial t} &= C D^T v + \frac{\partial \sigma^0}{\partial t} \end{aligned} \tag{43}$$

with

$$v = [v_x \;\; v_z]^T, \;\; \sigma = [\sigma_{xx} \;\; \sigma_{zz} \;\; \sigma_{xz}]^T \tag{44}$$

$$C = \begin{pmatrix} C_{11} & C_{13} & C_{14} \\ C_{13} & C_{33} & C_{34} \\ C_{14} & C_{34} & C_{44} \end{pmatrix}, \;\; D = \begin{pmatrix} \partial x_1 & 0 & \partial x_3 \\ 0 & \partial x_3 & \partial x_1 \end{pmatrix}. \tag{45}$$

2D elastic equations in a VTI medium

- In the VTI (Vertical Transverse Isotropic) approximation, C_{14} and C_{34} vanishes yielding

$$\begin{aligned} \rho \frac{\partial v}{\partial t} &= D\sigma + F \\ \frac{\partial \sigma}{\partial t} &= CD^T v + \frac{\partial \sigma^0}{\partial t} \end{aligned} \tag{43}$$

with

$$v = [v_x \ \ v_z]^T, \ \ \sigma = [\sigma_{xx} \ \ \sigma_{zz} \ \ \sigma_{xz}]^T \tag{44}$$

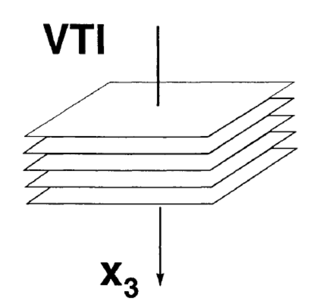
$$C = \begin{pmatrix} C_{11} & C_{13} & 0 \\ C_{13} & C_{33} & 0 \\ 0 & 0 & C_{44} \end{pmatrix}, \ \ D = \begin{pmatrix} \partial x_1 & 0 & \partial x_3 \\ 0 & \partial x_3 & \partial x_1 \end{pmatrix}. \tag{45}$$

- Expanding it we find (assuming zero stress source and only vertical and horizontal force sources)

$$
\begin{aligned}
&\frac{\partial v_x}{\partial t} - \frac{1}{\rho}\frac{\partial \sigma_{xx}}{\partial x} - \frac{1}{\rho}\frac{\partial \sigma_{xz}}{\partial z} = f_x \\
&\frac{\partial v_z}{\partial t} - \frac{1}{\rho}\frac{\partial \sigma_{xz}}{\partial x} - \frac{1}{\rho}\frac{\partial \sigma_{zz}}{\partial z} = f_z \\
&\frac{\partial \sigma_{xx}}{\partial t} - C_{11}\frac{\partial v_x}{\partial x} - C_{13}\frac{\partial v_z}{\partial z} = 0 \\
&\frac{\partial \sigma_{zz}}{\partial t} - C_{13}\frac{\partial v_x}{\partial x} - C_{33}\frac{\partial v_z}{\partial z} = 0 \\
&\frac{\partial \sigma_{xz}}{\partial t} - \frac{C_{44}}{2}\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) = 0
\end{aligned}
\tag{43}
$$

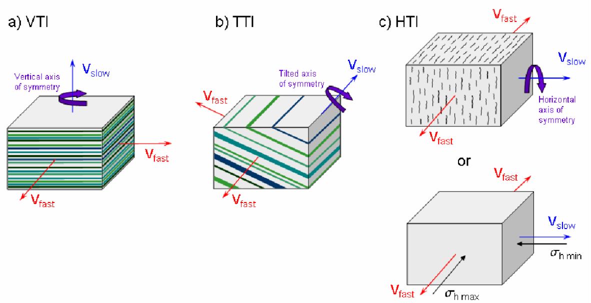
Vertical Transverse Isotropy

- A VTI medium is an effective medium which can be associated with the vertical piling of thin layers
- By thin we mean : layer thickness smaller than the wavelength of the propagated wave
- In this simple anisotropy case, the effective medium is homogeneous, with a horizontal velocity which is faster than the vertical velocity
- The wave are not reflected by the layers, but apparently, it travels faster in the horizontal (x, y) plane and slower in the vertical direction z



General Transverse Isotropy

- This can be extended to the case of horizontal layering (HTI) media
- More generally, this can be extended to any layering structure with an angle: we speak of TTI anisotropy in this case for Tilted Transverse Isotropy



Parameterization in a 2D VTI medium with Thomsen parameters

We can parameterize a 2D VTI medium with five independent parameters

- V_P : pressure wave velocity
- V_S : shear wave velocity
- ρ : density
- ϵ and δ : Thomsen parameter to represent VTI anisotropy (Thomsen, 1986)

The relations with the stiffness matrix coefficient are

$$\begin{aligned} C_{33} &= \rho V_P^2 \\ C_{11} &= \rho V_P^2(1+2\epsilon) \\ C_{44} &= \rho V_S^2 \\ C_{13} &= -\rho V_S^2 + \rho\sqrt{\left(v_P^2 - v_S^2\right)^2 + 2\delta v_P^2\left(v_P^2 - v_S^2\right)} \end{aligned} \tag{44}$$

Example of propagation in a 2D VTI medium

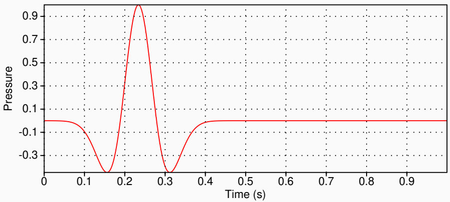
Source settings

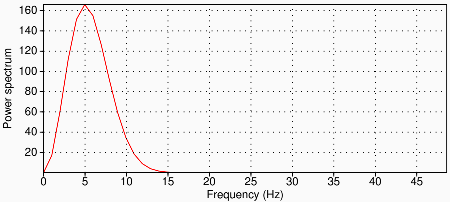
- Vertical force source f_z, localized in space (Diract) and with a Ricker wavelet time signature

$$f_z(x, z, t) = \delta(x - x_S)\delta(z - z_S)r(t) \tag{45}$$

with (x_S, z_S) the spatial coordinates of the source and

$$r(t) = \left(1 - 2\pi^2 f_0^2 (t - t_0)^2\right) e^{-\pi^2 f_0^2 (t - t_0)^2} \tag{46}$$





Example of propagation: homogeneous isotropic medium

Medium settings: homogeneous medium

First modeling in an isotropic medium

- $V_P = 2000$ m.s^{-1} : pressure wave velocity
- $V_S = 1000$ m.s^{-1} : shear wave velocity
- $\rho = 2000$ kg.m^{-3}: density
- $\epsilon = 0$: Thomsen parameter 1
- $\delta = 0$: Thomsen parameter 2

Example of propagation: homogeneous isotropic medium

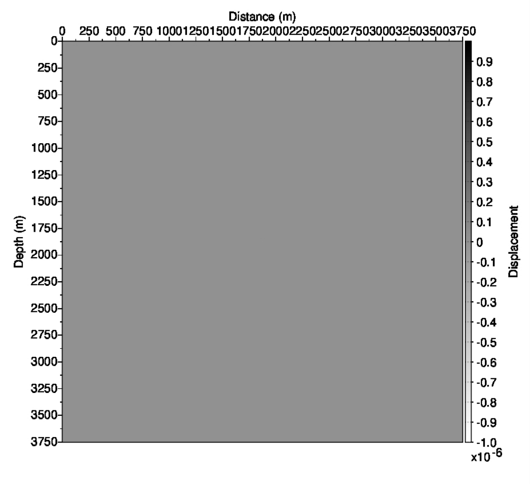


Figure 1: v_z field with a vertical force source emitting a 5 Hz Ricker wavelet in the center of an isotropic medium

Example of propagation: homogeneous isotropic medium

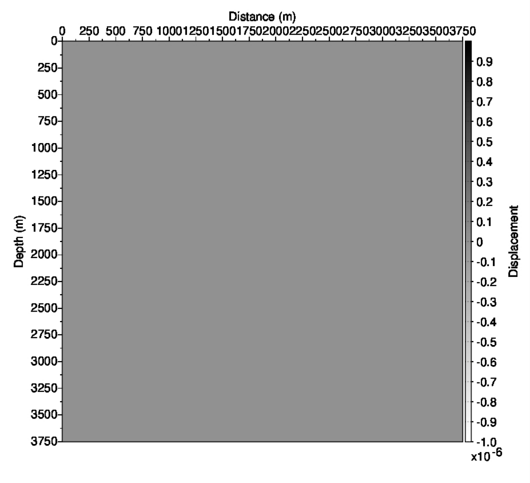


Figure 2: v_x field with a vertical force source emitting a 5 Hz Ricker wavelet in the center of an isotropic medium

Example of propagation: homogeneous VTI elliptic medium

Medium settings: homogeneous medium

Second modeling in an VTI elliptic medium

- $V_P = 2000$ m.s^{-1} : pressure wave velocity
- $V_S = 1000$ m.s^{-1} : shear wave velocity
- $\rho = 2000$ kg.m^{-3}: density
- $\epsilon = 0.3$: Thomsen parameter 1
- $\delta = 0.3$: Thomsen parameter 2

Example of propagation: homogeneous VTI elliptic medium

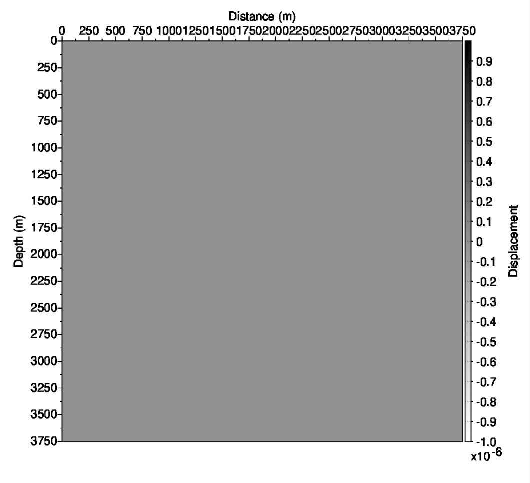


Figure 3: v_z field with a vertical force source emitting a 5 Hz Ricker wavelet in the center of a VTI elliptic medium

Example of propagation: homogeneous VTI anelliptic medium

Medium settings: homogeneous medium

Second modeling in an VTI anelliptic medium

- $V_P = 2000$ m.s^{-1} : pressure wave velocity
- $V_S = 1000$ m.s^{-1} : shear wave velocity
- $\rho = 2000$ kg.m^{-3}: density
- $\epsilon = 0.3$: Thomsen parameter 1
- $\delta = 0.1$: Thomsen parameter 2

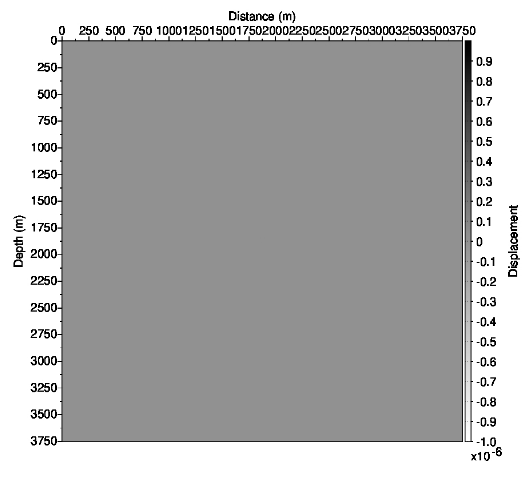
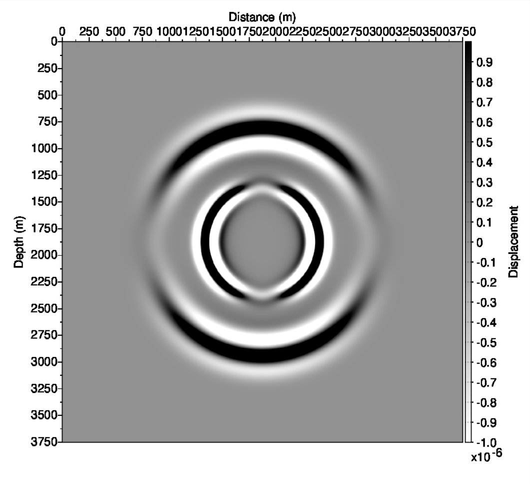


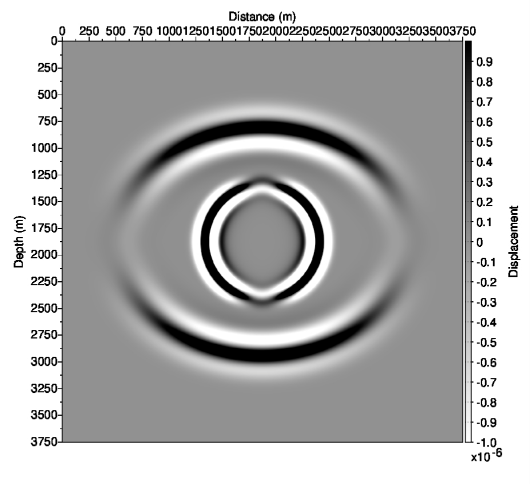
Figure 4: v_z field with a vertical force source emitting a 5 Hz Ricker wavelet in the center of a VTI anelliptic medium

Comparison between isotropic and VTI media



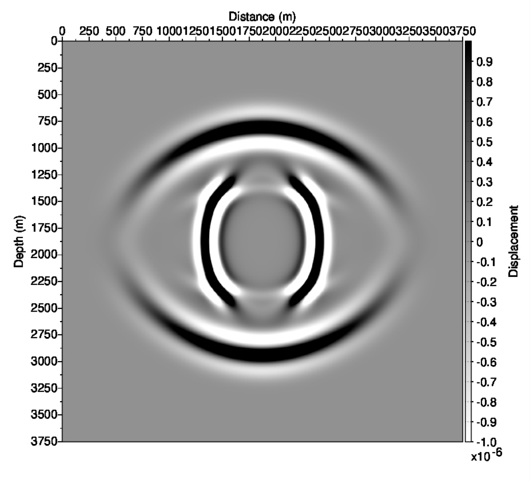
v_z in an isotropic medium

Comparison between isotropic and VTI media



v_z in an elliptic VTI medium

Comparison between isotropic and VTI media



v_z in an anelliptic VTI medium

Choice of boundary conditions

The choice of homogeneous Dirichlet boundary conditions corresponds to

$$\begin{aligned} \rho \frac{\partial v}{\partial t} &= D\sigma + F \\ \frac{\partial \sigma}{\partial t} &= CD^T v + \frac{\partial \sigma^0}{\partial t} \end{aligned} \tag{47}$$

with

$$\begin{cases} v(x,t) = 0; & x \in \partial\Omega, \\ \sigma(x,t) = 0, & x \in \partial\Omega \end{cases} \tag{48}$$

In terms of simulation, this is what it yields

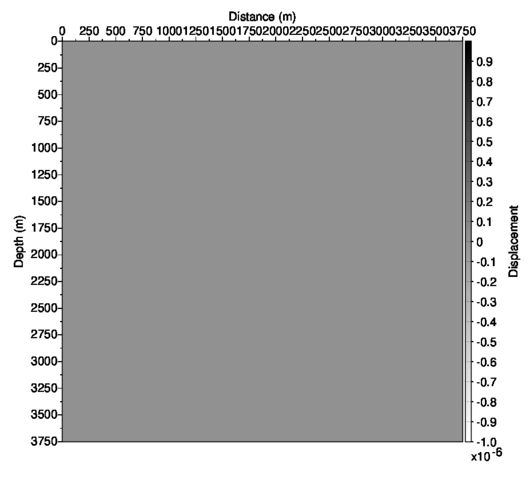


Figure 5: v_z field with a vertical force source in the center of a VTI isotropic medium emitting a 5 Hz Ricker

However, what we would like resembles more to

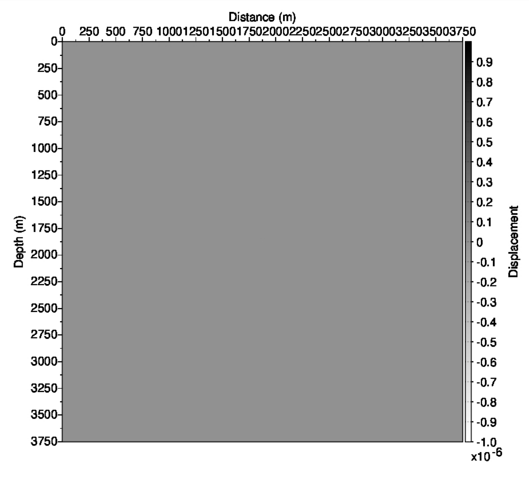


Figure 6: v_z field with a vertical force source in the center of a VTI isotropic medium emitting a 5 Hz Ricker

Choice of boundary conditions

How to design such absorbing boundary conditions?

References

Thomsen, L. A. (1986). Weak elastic anisotropy. *Geophysics*, 51:1954–1966.