Analysis (and geometry) of alternating projection algorithms

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based on joint work with Adrian Lewis and Russell Luke
Outline

1. Alternating convex projections
2. Nonconvex projections
3. Alternating nonconvex projections
4. Regularity and conditionning
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1. Alternating convex projections
2. Nonconvex projections
3. Alternating nonconvex projections
4. Regularity and conditionning
Projection, distance and convexity

In a Euclidean space \((\mathbb{R}^n, \| \cdot \|)\)

For closed \(M \subset \mathbb{R}^n\), the distance of \(x\) from \(M\)

\[
d_M(x) = \min\{\|x - y\| : y \in M\}
\]

and the projection of \(x\) onto \(M\)

\[
P_M(x) = \arg\min\{\|x - y\| : y \in M\}
\]

If \(M\) is convex, \(P_M(x)\) is singleton. Otherwise, it is not for some \(x\) for sure!
Alternating projections on subspaces

For affine subspaces $M$ and $N$, von Neumann '33 studied:

$$(P_M P_M)^n(x) \rightarrow P_M \cap N(x)$$
Alternating projections on subspaces

For affine subspaces $M$ and $N$, von Neumann '33 studied:

$$(P_M P_M)^n(x) \longrightarrow P_M \cap N(x)$$

Convergence is linear at rate $(\cos \theta)^2$, indeed:

$$\|(P_M P_M)^n(x) - P_M \cap N(x)\| \leq (\cos \theta)^{2n-1}\|x\|$$

where $\theta$ is the angle between $M$ and $N$ (Aronszajn '50)
Alternating convex projections

Alternating projections naturally extends to closed convex sets $M$ and $N$

Bregman '65 proves: $\left(P_M P_M\right)^n(x) \rightarrow M \cap N$
Alternating convex projections

Alternating projections naturally extends to closed convex sets $M$ and $N$

Bregman '65 proves: $\lim_{n \to \infty}(P_M P_M)^n(x) \to M \cap N$

Convergence is linear providing $M \cap \text{int } N \neq \emptyset$ (more generally $\text{ri } M \cap \text{ri } N \neq \emptyset$)

(Polyak et al '67, Bauschke-Borwein '93)
Algorithm simple but powerful

To find a point \( x \in M \cap N \), with \( M \) and \( N \) closed convex sets on \( \mathbb{R}^n \)
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Alternating convex projections is a basic algorithm

\[ \rightarrow \] many enhancements, among them:

- in Hilbert, complex spaces...
- several sets, averaged, cyclic projections...
- relaxations, regularization (under/over-relaxed, AAR, Dykstra,...)
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Applications: statistics, finance, engineering sciences, image processing...
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Ex2: Xu-Zikatanov ’02 draw connection alternating projection with methods of PDE’s (domain decomposition or multigrid methods)
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Ex2: Xu-Zikatanov '02 draw connection alternating projection with methods of PDE’s (domain decomposition or multigrid methods)

Ex3: Combettes et al. '10 “on the effectiveness of projection methods for convex feasibility problems with linear inequality constraints”
Example in Finance

For symmetric matrix $C$, computing the nearest correlation matrix: computing the projection of $C$ onto the intersection of $S_n^+$ the semidefinite positive matrices, and the matrices with ones the diagonal.
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How to compute the nearest correlation matrix:

- alternating projection (Higham ’02) (+ Dykstra correction)
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How to compute the nearest correlation matrix:

1. alternating projection (Higham ’02) (+ Dykstra correction)

2. Lagrangian duality (Malick ’04) → efficient algorithm
Nonconvex heuristic

alternating convex projections is a good method...
Nonconvex heuristic

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...alternating nonconvex projections is also a popular heuristic!
Nonconvex heuristic

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Examples

- Optics: phase retrieval of images (Combettes et al.’02)
  Simple version: given $a_j \in \mathbb{C}^k$, find $x \in \mathbb{C}^k$, so
  $$|\langle a_j, x \rangle| = b_j \quad (j = 1, \ldots, m)$$
  with alternative projections onto
  $$M = \{(x, z) \in \mathbb{C}^n \times \mathbb{C}^m : Ax = z\}$$
  $$N = \{(x, z) : |z_j| = b_j, (j = 1, \ldots, m)\}$$
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    \]
- Control: low-order control design (eg Grigoriadis-Beran '00)
  affine \( M \subset \{n\text{-by-}n \text{ symmetric matrices}\} \)
  \[
  N = \{\text{positive semidefinite matrices of rank } r\}
  \]
Numerical illustration

Find a 100-by-110 matrix $X$ of rank 4, satisfying 450 equations

$$A(X) = b$$

(simple analogue of the low-rank control problem)
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→ alternating projections onto

$$M = \{X : A(X) = b\} \quad \text{(by inverting } AA^\top\text{)}$$

$$N = \{X : \text{rank}(X) = 4\} \quad \text{(via singular value decomposition)}$$
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$\rightarrow$ alternating projections onto $M = \{X : A(X) = b\}$ (by inverting $AA^\top$)

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Alternating nonconvex projections

Why does alternating nonconvex projections work?
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Projecting onto nonconvex sets ??
How does linear convergence appear?
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Projecting onto nonconvex sets??:
How does linear convergence appear?

Few answers:

- 1st results in *Combettes-Trussel '90* (with convex-like techniques)
- linear cvg in special cases (*Orsi '06* for a matrix analysis pb),...
- or of special algos (*Attouch-Bolte-Redont-Soubeyrand '08, Luke '09*)
- *Lewis-Malick '07, Lewis-Luke-Malick '08* – whose ingredients are:
  - geometry of the intersection and of the algorithm
  - interpretation with nonsmooth analysis
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In this talk:

1. Easy-to-compute nonconvex projections
2. Convergence of the algorithm through nice geometry
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Easy nonconvex projections

For closed nonconvex $M \subset \mathbb{R}^n$, the projection $P_M(x)$ is somewhere non-singleton. But projection may still be easy...
Easy nonconvex projections

For closed nonconvex \( M \subset \mathbb{R}^n \), the projection \( P_M(x) \) is somewhere nonsingleton. But projection may still be easy...

Examples

- Single quadratic constraint

\[
M = \left\{ x \in \mathbb{R}^n : \ x^\top A x + b^\top x = c \right\}
\]

Projection is analogous to trust-region subproblems, solvable with a special Newton method
Easy nonconvex projections

For closed nonconvex $M \subset \mathbb{R}^n$, the projection $P_M(x)$ is somewhere nonsingleton. But projection may still be easy...

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- **Single quadratic constraint**
  
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- **Rank constraint**

  $$M = \left\{ X \in \mathbb{R}^{n \times m} : \text{rank}(X) = r \right\}$$

  To project, find a singular value decomposition $X = UDV$ and zero all but the first $r$ largest singular values in $D$
Spectral sets

For permutation-invariant $K \subset \mathbb{R}^n$, the spectral set of symmetric matrices

$$\lambda^{-1}(K) = \{X \in S_n : (\lambda_1(X), \ldots, \lambda_n(X)) \in K\}$$

(analogous definition for sets of matrices described by singular values)
Spectral sets

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Examples

- $K = \mathbb{R}_+^n$ gives the positive semidefinite cone $S_n^+$
- $K = \Sigma_n \cdot x$ gives an isospectral set (given $x$)
- $K = \{ x : \text{Card} (\text{argmax}\{x_i\}) = r \}$ gives $\{ X : \lambda_{\max}(X) \text{ with } r \}$
Easy spectral projections

The following result (Lewis-Malick '07) generalizes previous (partial) results about projections onto some spectral sets (eg Higham '88, Oustry '02)

**Theorem (projection onto spectral sets)**

If \( y \in P_K(x) \) and \( U \) orthogonal, then

\[
U^\top (\text{Diag } y) U \in P_{\lambda^{-1}(K)}(U^\top (\text{Diag } x) U)
\]
Prox-regular spectral sets

Let’s take the chance to say more on spectral sets

Transfer of structure: if $K$ is invariant by permutation of entries
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Lewis ’96: $K$ convex $\implies \lambda^{-1}(K)$ convex
Prox-regular spectral sets

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Transfer of **structure**: if $K$ is invariant by permutation of entries

1. **Lewis ’96**: $K$ convex $\implies \lambda^{-1}(K)$ convex
2. **Daniilidis-Lewis-Malick-Sendov ’08** proves

$$K \text{ prox-regular} \implies \lambda^{-1}(K) \text{ prox-regular}$$
Prox-regular spectral sets

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Transfer of structure: if \( K \) is invariant by permutation of entries

1. Lewis ’96: \( K \) convex \( \implies \lambda^{-1}(K) \) convex
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\[ K \text{ prox-regular} \implies \lambda^{-1}(K) \text{ prox-regular} \]

General notion of prox-regularity (eg Rock.-Poliquin-Thibault ’00):

\[ P_M \text{ is locally unique} \]

\( \rightarrow \) prox-regular spectral sets have locally all the good properties!

(Ex: manifolds...)

\[ M \]
Prox-regular spectral sets in practice

Many spectral sets in alternative nonconvex projections

Numerical algebra: nonnegative inverse eigenvalue pb (Orsi '06)
For $\bar{\lambda}$ given, find $X \in M \cap N$

\[
M = \{ X \in \mathbb{R}^{n \times n} : \lambda(X) = \bar{\lambda} \}
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N = \{ X \in \mathbb{R}^{n \times n} : X_{ij} \geq 0 \}
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2 Image processing: design of tight frames (Tropp et al '05)
Find the associated Gram matrix $X \in M \cap N$

$$M = \{ X \in \mathbb{C}^{n \times n} : \lambda(X) = (n, \ldots, n, 0, \ldots, 0)/d \}$$

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— Many other types of sets (ex: Phase retrieval)...
Last word on spectral sets

If $M$ is smooth manifold, is $\lambda^{-1}(M)$ smooth as well?
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- No, in general
  Ex: $M = ] - 1, 1[ \times \{0\} \subset \mathbb{R}^2$, and $\lambda^{-1}(M)$ has a kink

- Yes, if $M$ is loccally symmetric!
  A neigbd of $M$ around $x \in M$ is invariant under permutations $\sigma$ such that $\sigma x = x$

Moreover, we know the dimension of $\lambda^{-1}(M)$

Not straightforward... 43 pages of sheer joy (Daniilidis-Malick-Sendov '09)
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Alternating nonconvex projections

Theorem (local linear convergence)

For closed sets $M, N \subset \mathbb{R}^n$. Assume

- **strong regularity** holds at $\bar{x} \in M \cap N$
- $M$ is **super-regular** at $\bar{x}$
- *initial* $x_0$ near $\bar{x}$

Then alternating projection method converges $R$-linearly to $M \cap N$
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- More in Lewis-Luke-Malick ’08...
Strong regularity

Simple definition:

\[ N_M(\bar{x}) \cap -N_N(\bar{x}) = \{0\} \]

In other words, the minimal angle between \( N_M(\bar{x}) \) and \(-N_N(\bar{x})\) is \( \theta > 0 \)
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Strong regularity is a standard notion of nonsmooth analysis (see eg Kummer ’06), useful in theory (ex: normal cone to the intersection)
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Examples

1. The intersection of two smooth manifolds is strongly regular \( \iff \) the manifolds are transverse
2. The intersection of two convex sets is strongly regular \( \iff \) no separating hyperplane
Super-regularity

Notion (not standard!) introduced in Lewis-Luke-Malick '08

Examples of super-regular sets

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2. smooth manifolds
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prox-regular \( \subset \) super-regular \( \subset \) (Clarke) regular
Sketch proof

The geometry controls the asymptotical improvement:

For \( x \in M \) near \( \bar{x} \),

\[
\frac{\| P_M P_N(x) - P_N(x) \|}{\| P_N(x) - x \|} \approx \cos \theta
\]

is not much larger than \( \cos \theta \)
Consequence for averaged projections

Method of averaged projections

\[ z_M \in P_M(x) \]
\[ z_N \in P_N(x) \]
\[ x \leftarrow \frac{1}{2} (z_N + z_M) \]
Consequence for averaged projections

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Corollary (linear convergence of averaged projections)

For any closed \( M, N \subset \mathbb{R}^n \), if strong regularity holds at \( \bar{x} \in M \cap N \), then starting with \( x_0 \) near \( \bar{x} \), averaged projections converge linearly to \( M \cap N \)
More on averaged projections

**Proof:** Following Auslender ’69 in the convex case, just consider alternating projections in $\mathbb{R}^n \times \mathbb{R}^n$ between

$$M \times N \quad \{(x, x) : x \in \mathbb{R}^n\} \text{ (super-regular)}$$
More on averaged projections

1 Proof: Following Auslender ’69 in the convex case, just consider alternating projections in $\mathbb{R}^n \times \mathbb{R}^n$ between

$$M \times N \quad \{(x, x) : x \in \mathbb{R}^n\} \quad \text{(super-regular)}$$

2 Interpretation as minimization (Lewis-Luke-Malick ’08)

If $M$ and $N$ both prox-regular, averaged projections is just the steepest descent (with unit step size) applied to

$$f(x) = \frac{1}{4} \left( d_M^2(x) + d_N^2(x) \right)$$

$\longrightarrow$ Q-linear convergence: improvements at each iteration

$$\frac{f(x_{k+1})}{f(x_k)} < 1 - \frac{1}{2\kappa^2}$$
Averaged projections to find \( d \)-by-\( m \) matrix \( U \in L \cap M \cap C \)

- (linear) \( L = \{ U \in \mathbb{R}^{d \times m} : U = PW \} \)
- (smooth) \( M = \{ U \in \mathbb{R}^{d \times m} : UU^\top = I \} \)
- (convex) \( C = \{ U \in \mathbb{R}^{d \times m} : \|U\|_\infty \leq \alpha \} \)
Numerical illustration

Averaged projections to find $d$-by-$m$ matrix $U \in L \cap M \cap C$

(linear) \hspace{1cm} L = \{U \in \mathbb{R}^{d \times m} : U = PW\}

(smooth) \hspace{1cm} M = \{U \in \mathbb{R}^{d \times m} : UU^\top = I\}

(convex) \hspace{1cm} C = \{U \in \mathbb{R}^{d \times m} : \|U\|_\infty \leq \alpha\}

\[
\frac{f(U_{k+1})}{f(U_k)} \leq 0.96 < 1
\]

where $f$ is the sum of the squared distances
1. Alternating convex projections

2. Nonconvex projections

3. Alternating nonconvex projections

4. Regularity and conditionning
alternating projections + small angle

Weak error bound:

\[ d^2_{M \cap N}(\cdot) \leq \rho \left( d^2_M(\cdot) + d^2_N(\cdot) \right) \]

needs \( \rho \) large
alternating projections + small angle

Weak error bound:

\[ d_{M \cap N}^2(\cdot) \leq \rho \left( d_M^2(\cdot) + d_N^2(\cdot) \right) \]

needs \( \rho \) large

Small perturbations render the problem ill-posed
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Alternating method converges with slow linear rate \( \cos \theta \)
Conditionning

For linear subspaces for example, it is obvious on picture that instances are not well-conditionning, when the angle $\theta$ is small...

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- Demmel paradigm (Demmel '87)
- metric regularity (eg Rockafellar-Wets '02)
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$\rightarrow$ alternating projections turns out be a nice illustration of
  - Demmel paradigm (Demmel ’87)
  - metric regularity (eg Rockafellar-Wets '02)

For many computational problems, three equivalent properties characterize “hard” instances:
  1. a posteriori error bounds are weak
  2. the distance to an ill-posed instance is small
  3. basic algorithms are slow
Consider the positive-definite linear system

$$Ax = y$$

\[ \|x - A^{-1}y\| \leq \frac{1}{\lambda_{\text{min}}(A)} \|Ax - y\| \]

a posteriori error bound is weak
Simple example: solving linear system

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3. Basic algorithms converge with slow linear rate. Eg:

\[
\left( \frac{\kappa - 1}{\kappa + 1} \right)^2 \quad \text{and} \quad \left( \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^2
\]

for steepest descent and conjugate gradients, where

\[ \kappa = \lambda_{\text{max}}(A)/\lambda_{\text{min}}(A) \]
A very general framework: inversion

Given set-valued $F: \mathbb{R}^n \mapsto \mathbb{R}^p$, suppose

- $F(x)$ is easy to compute
- $F^{-1}(x)$ is hard to compute

**Problem:** Given a point $\bar{y}$, find some point $\bar{x} \in F^{-1}(\bar{y})$
General framework

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**Example:** (Lewis-Luke-Malick '08)
Given $M, N \subset \mathbb{R}^n$, find $\bar{x} \in M \cap N$
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**Example:** (Lewis-Luke-Malick ’08)
Given $M, N \subset \mathbb{R}^n$, find $\bar{x} \in M \cap N$
Define $G: \mathbb{R}^n \Rightarrow \mathbb{R}^n \times \mathbb{R}^n$ by

$$G(x) = (M - x) \times (N - x)$$

Finding $\bar{x} \in G^{-1}(0, 0) = M \cap N$ is solving the generalized equation

$$(0, 0) \in G(x)$$
To quantify local error bounds

**Definition (Metric regularity)**

Suppose $\bar{y} \in F(\bar{x})$. We say that $F$ is **metrically regular** at $(\bar{x}, \bar{y})$ if the local error bound

$$d_{F^{-1}}(y)(x) \leq \rho \ d_F(x)(y) \quad \text{for all } (x, y) \text{ near } (\bar{x}, \bar{y})$$

holds for some $\rho$. The **modulus** $\text{reg} \ F(\bar{x}, \bar{y})$ is the infimum of such $\rho$. 
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For alternating projections (ie: \((0, 0) \in G(x)\))

- \(G\) is metrically regular at \((\bar{x}, (0, 0))\)
  \[\iff\] closed \(M\) and \(N\) have strongly regular intersection
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For alternating projections (ie: $(0, 0) \in G(x)$)

- $G$ is metrically regular at $(\bar{x}, (0, 0))$
  - $\iff$ closed $M$ and $N$ have strongly regular intersection
- With $\theta$ the minimum angle between $N_M(\bar{x})$ and $-N_N(\bar{x})$
  - $\text{reg} \, G(\bar{x}, (0, 0)) = \frac{1}{\sqrt{1 - \cos \theta}}$

hence the modulus controls local linear convergence rates.
Modulus of metric regularity quantifies hardness of instances:

1. the a posteriori error bounds (definition of $\text{reg}$)
2. the distance to ill-posedness is $1/\text{reg}$ by a general theorem (Dontchev-Lewis-Rockafellar '03)
3. the rate of convergence of basic algorithms is governed by $\text{reg}$
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Illustrations of this quantified Demmel’s paradigm:

- linear systems
- alternating projections (Lewis-Malick '07, Lewis-Luke-Malick '08)
Condition number

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Illustrations of this quantified Demmel’s paradigm:

- linear systems
- alternating projections (Lewis-Malick '07, Lewis-Luke-Malick '08)
- proximal point methods (Aragon-Artacho-Dontchev-Geoffroy '05)
- several conceptual algorithms (Klatte-Kummer '07)
- errors bounds and descent methods (Luo-Tseng '93)
- more ?...
Summary

- Nonconvex projections are tractable in some usual situations.
- Alternating nonconvex projections is a tempting natural heuristic, often converges linearly, and is thus popular!
- The linear rate reflects the “condition number”

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\text{distance ill-posedness} \leftrightarrow \text{error bound} \leftrightarrow \text{rate}
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Alternating projections on manifolds

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thanks!