Applications of SDP least-squares in finance and combinatorics

Jérôme MALICK

CNRS, Lab. J. Kuntzmann, Grenoble

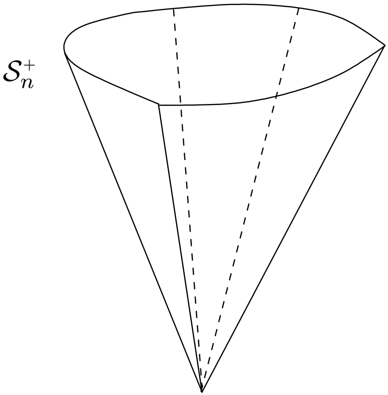
CORE math. prog. seminar – 11 March 2008

Outline

1. Semidefinite programming and least-squares
2. Appli 1: constructing covariance matrices in finance
3. Appli 2: relaxing binary quadratic problems
4. Appli 3: regularizing methods for solving SDP

Outline

Semidefinite programming



The cone of positive semidef. matrices

$$\begin{aligned}\mathcal{S}_n^+ &= \{A \in \mathcal{S}_n, \ \ \forall x \in \mathbb{E}, \ x^\top A x \geqslant 0\} \\ &= \{A \in \mathcal{S}_n, \ \ \lambda_{\min}(A) \geqslant 0\}\end{aligned}$$

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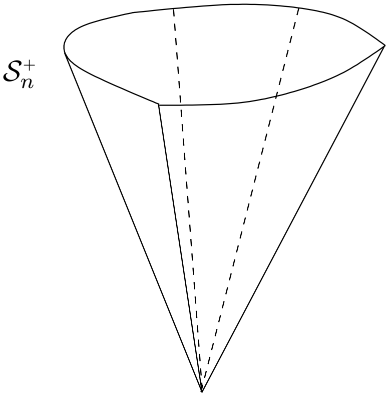

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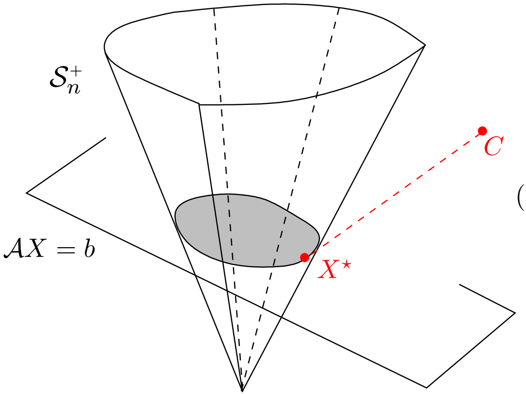
Standard semidefinite programming

$$(\mathrm{SDP}) \left\{ \begin{array}{ll} \min & \langle C, X\rangle \\ & \mathcal{A}X = b \\ & X \succcurlyeq 0 \end{array} \right.$$

R. Saigal, L. Vandenberghe, and H. Wolkowicz
Handbook of Semidefinite Programming
Kluwer, 2000

Semidefinite least-squares



$$\text{(SDLS)} \begin{cases} \min \quad \|X - C\|^2 \\ \quad \mathcal{A}X = b \\ \quad X \succcurlyeq 0 \end{cases}$$

SDLS problems

Example: compute the nearest correlation matrix

$$\left\{\begin{array}{l} \min \quad \|X - C\|^2 \\ \quad \operatorname{diag}(X) = e \\ \quad X \succcurlyeq 0 \end{array}\right.$$

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In general: general form tackled

$$\begin{cases} \min & \langle C', X\rangle + \|W(X - C)W\|^2 \\ & \mathcal{A}X = b \\ & \mathcal{A}'X \leqslant b' \\ & X \succcurlyeq 0 \end{cases}$$

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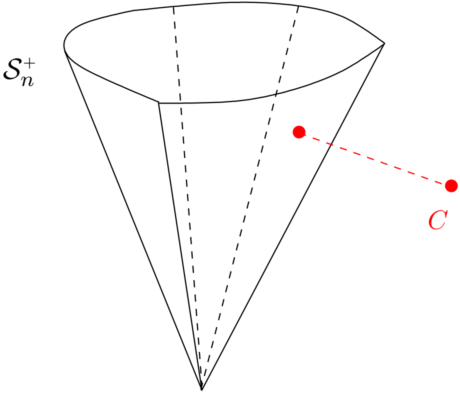
Question: How to solve semidefinite least-squares ?

Basic fact: Project onto the cone



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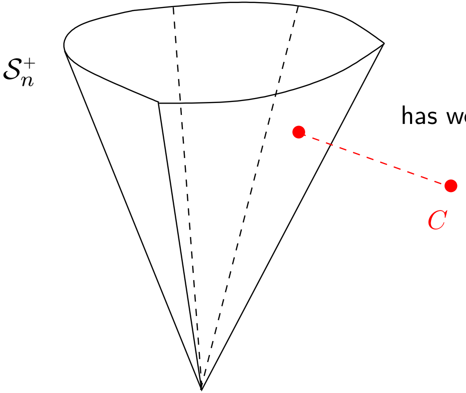

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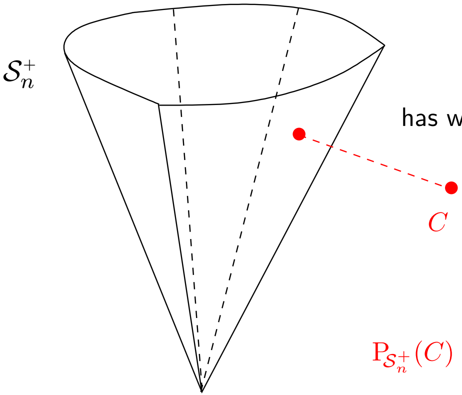

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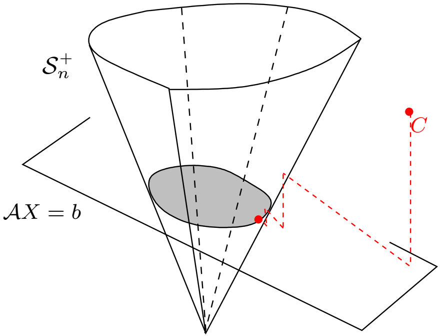
$$\mathrm{P}_{\mathcal{S}_n^+}(C) = P \begin{bmatrix} \max\{0, \lambda_1\} & & \\ & \ddots & \\ & & \max\{0, \lambda_n\} \end{bmatrix} P^\top$$

1st idea: alternating projections



Projection onto the intersection of two (convex) sets

$\longrightarrow$ alternating projections
$+$ Dykstra correction

N. Higham
Computing nearest symmetric correlation matrix - a problem from finance
IMA Journal of Numerical Analysis, 2002.

Next ideas

- 2nd idea: reformulate SDLS as a linear conic program

$$\text{(SDLS)} \iff \begin{cases} \min \quad t \\ \quad \|X - C\| \leqslant t \\ \quad \mathcal{A}X = b, \ X \succcurlyeq 0 \end{cases}$$

with usual conic solvers (Sedumi, SDPT3,...) $\longrightarrow$ no good results

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- 4th idea: by duality (Lagrangian duality + dual resolution)

J. Malick
A dual approach to semidefinite least-squares problems
SIAM Journal on Matrix Analysis and Applications, 2004

Lagrangian duality

Apply standard machinery:

$$L(X;\lambda) = \|X - C\|^2 - \lambda^\top(\mathcal{A}X - b)$$

$$\theta(\lambda) = \begin{cases} \min & L(X;\lambda) \\ X \succcurlyeq 0 \end{cases}$$

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$$\text{(dual)} \begin{cases} \max & \theta(\lambda) \\ \lambda \in \mathbb{R}^m \end{cases}$$

dual problem is convex and differentiable !

Solving the dual problem

Apply standard algorithms:

$$(\text{dual}) \begin{cases} \max & \theta(\lambda) \\ & \lambda \in \mathbb{R}^m \end{cases}$$

variable metric descent

$$\lambda_{k+1} = \lambda_k - \tau_k W_k \nabla\theta(\lambda_k)$$

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3. Generalized Newton

$$W_k = [H_k]^{-1} \qquad \text{with } H_k \in \partial_c^2\theta(\lambda_k)$$

Numerical example

Compute the nearest correlation matrix (for random C of various sizes)

$$\begin{cases} \min \quad \|X - C\|^2 \\ \quad \operatorname{diag}(X) = e \\ \quad X \succcurlyeq 0 \end{cases}$$

Stop if relative error

$$\frac{\| \operatorname{diag}(X) - e\|}{\sqrt{n}} \leqslant 10^{-7}$$

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size	cpu time	nb iter.
100	0.2 s	14
500	16.3 s	17
1000	2 min 05 s	18
2000	17 min 41 s	19
3000	1h 08 min 14 s	19

Conclusion: SDP vs. SDLS

- Semidefinite programming

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$\longrightarrow$ SDLS are easier than SDP !

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- SDLS modelling ? much less than SDP, but

  $\longrightarrow$ in next applications, SDLS problems naturally appear...

Outline

Estimating covariance matrices

- Random variable $X \in \mathbb{R}^n$ with unknown moments
  (Eg: returns of n assets)
- Observations of X : $x_t \in \mathbb{R}^n$ for $t = 1, \ldots, T$
  (Eg: historical charts of the assets)
- Empirical estimations

$$\mu = \frac{1}{T}\sum_{t=1}^{T} x_t \qquad \widetilde{\Gamma} = \frac{1}{T}\sum_{t=1}^{T}(x_t - \mu)(x_t - \mu)^\top \succcurlyeq 0$$

- Beyond statistics:

$\longrightarrow$ in practice, $\widetilde{\Gamma}$ has weaknesses
(Eg: incomplete data, choice of T, bad conditioning,...)

$\longrightarrow$ how to add exogenous (nonhistorical) information ?
(Eg: set values to entries, impose structure...)

Covariance matrices in finance

Goal: to construct "good" covariance matrices with special structure, from the empirical estimation $\widetilde{\Gamma}$

Many uses in quantitative finance, eg:

- Portfolio selection
- Correlation Stress testing
- Convexification of BGM model
- Aggregation to a global risk model from local models and cross-market estimates

RaisePartner Corp.

http://www.raisepartner.com

Example: portfolio selection

Find the portfolio $x = (x_1, \ldots, x_n)$ with minimum risk, as the solution of

$$\begin{cases} \min \quad x^\top \Gamma x \\ \quad x^\top \rho \geqslant r_0 \\ \quad x \in \Delta \end{cases}$$

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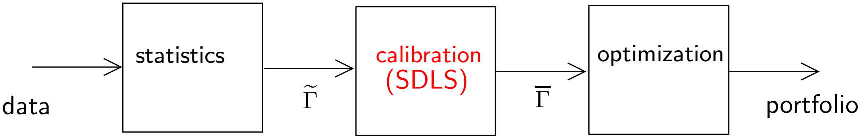

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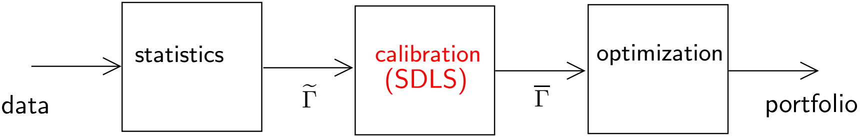
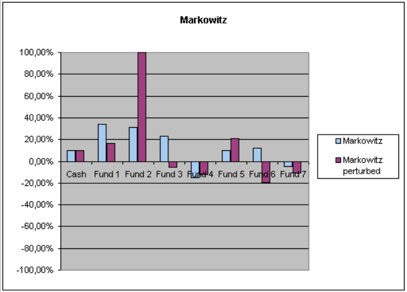


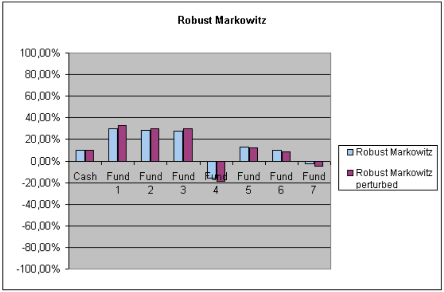
Preliminary calibration: in selection problem, use $\bar{\Gamma}$ the solution of

$$\begin{cases} \min \quad \|X - \widetilde{\Gamma}\|^2 \\ \quad X - \alpha I_n \succcurlyeq 0 \\ \quad \operatorname{trace}(X) = \operatorname{trace}(\widetilde{\Gamma}) \\ \quad \langle A_i, X \rangle \in [a_i, b_i] \end{cases}$$

Simple illustration

	Cash	Fund 1	Fund 2	Fund 3	Fund 4	Fund 5	Fund 6	Fund 7	Total
Markowitz	10,00%	33,79%	30,81%	23,32%	-14,92%	9,66%	12,13%	-4,78%	100,00%
Markowitz Perturbed	10,00%	16,61%	100,00%	-5,33%	-11,97%	21,15%	-19,61%	-10,85%	100,00%
Robust Markowitz	10,00%	30,05%	28,34%	27,73%	-16,21%	12,90%	10,10%	-2,91%	100,00%
Robust Markowitz Perturbe	10,00%	32,72%	29,97%	29,91%	-18,55%	12,39%	8,35%	-4,79%	100,00%





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Bounds, relaxations

Bounding is a basic tool of combinatorial optimization
If we can't

$$\begin{cases} \max & \text{(easy objective function)} \\ x \in & \text{(difficult constraints)} \end{cases}$$

we are content with an upper bound, computing after relaxation

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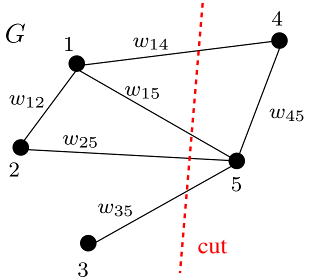
Enlarging the feasible set $\longrightarrow$ $\begin{cases} \max & \text{(easy fonction)} \\ x \in & \text{(easy constraints)} \end{cases}$

Example: max-cut

$G = (V, E)$ undirected graph
$Q = (-w_{ij})_{ij}$ weight-matrix

finding the cut of maximal weight

$$= \begin{cases} \max & x^\top Q\, x \\ & x \in \{-1,1\}^n \end{cases}$$



$\longrightarrow$ NP hard

$\longrightarrow$ Classical relaxations

- linear relaxation: $x_i \in \{-1,1\} \rightarrow x_i \in [-1,1]$
- SDP relaxation...
- many others !

SDP relaxation

$$(\text{max-cut}) \left\{ \begin{array}{ll} \max & x^\top Q\, x \\ & x \in \{-1,1\}^n \end{array} \right.$$

$$\boxed{X = xx^\top}$$

$$(\text{max-cut}) \left\{ \begin{array}{ll} \max & \langle Q, X\rangle \\ & X_{ii} = 1 \\ & X \succcurlyeq 0 \\ & \operatorname{rang} X = 1 \end{array} \right.$$

L. Lovász
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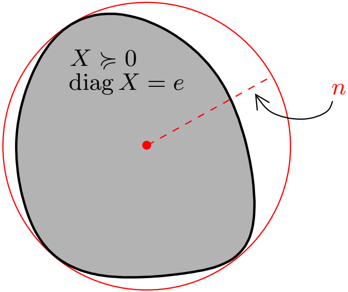
Spherical constraint

For all $X \succcurlyeq 0$ satisfying $X_{ii} = 1$, we have

$$\|X\| \leqslant n$$



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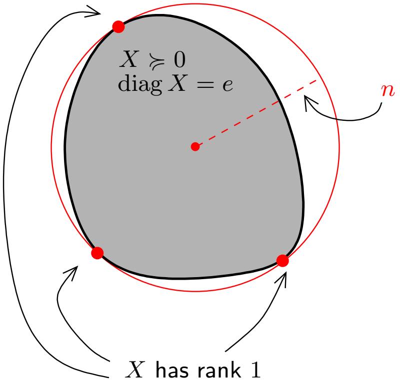
$$\|X\| = n \iff \operatorname{rang} X = 1$$

"spherical constraint"



Dualizing spherical constraint

$$(\text{max-cut}) \begin{cases} \max & \langle Q, X\rangle \\ & X_{ii} = 1 \quad \text{for all } i \\ & X \succcurlyeq 0 \\ & \|X\|^2 - n^2 = 0 \end{cases}$$

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$$L(X;\alpha) = \langle Q, X\rangle - \alpha(\|X\|^2 - n^2)$$ Lagrangian

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Lagrangian

$$\theta(\alpha) = \begin{cases} \max & L(X; \alpha) \\ & X_{ii} = 1 \quad \text{for all } i \\ & X \succcurlyeq 0 \end{cases}$$

dual function

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$$\theta(\alpha) = \text{Cste}_\alpha - \alpha \begin{cases} \min & \frac{1}{2}\|X - Q/\alpha\|^2 \\ & X_{ii} = 1 \quad \text{for all } i \\ & X \succcurlyeq 0 \end{cases}$$

$\longrightarrow$ SDLS (if $\alpha > 0$)

Bounds and duality gap

What does it yield ?

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- if $\alpha = 0$, there holds

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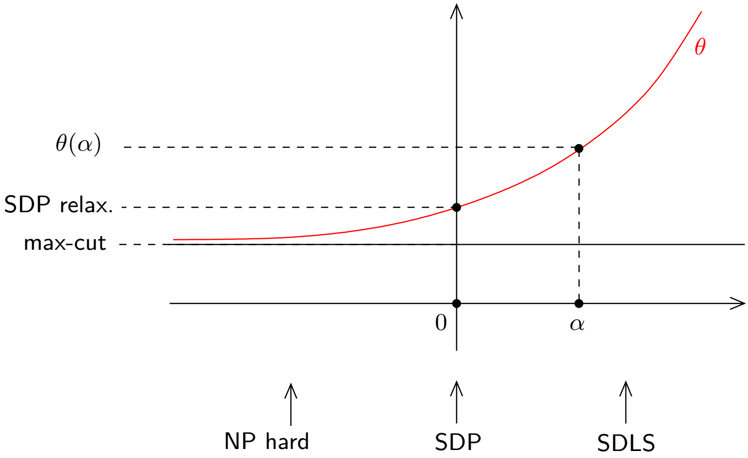
- if $\alpha < 0$ large, computing $\theta(\alpha)$ is NP hard !

No duality gap



Numerical example

For a random max-cut (500 nodes and 70% density) compute SDLS bounds $\theta(\alpha)$ for several $\alpha \in [0, 1]$

α	error	iters	time
1	94%	9	28 s
0.1	52%	16	53 s
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$\longrightarrow$ when $\alpha \downarrow 0$:

SDLS bounds get tighter (but always less than SDP one !)

SDLS problems become more difficult

Conclusion: SDLS bounds

Spherical constraint:

$\longrightarrow$ new formulation of rank-one constraint

$\longrightarrow$ new interpretation of SDP relaxation

$\longrightarrow$ opens the way for SDLS solvers

$\longrightarrow$ for any binary quadratic problem

J. Malick
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Current research:

⟶ more complete numerical experiments

⟶ good balance between accuracy and speed of computation ?

⟶ difficult combinatorial problems (k-cluster, QAP)

Outline

How to solve general SDP

1. primal-dual interior point methods
2. spectral bundle methods, $\lambda_{\min}(X)$
3. with low-rank methods, $X = RR^\top$ with $R \in \mathbb{R}^{n \times r}$
4. and others ! Sorry for not citing them...

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$\longrightarrow$ Joint work F. Rendl, J. Povh and A. Wiegele

Proximal approach on paper

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Introduce the convex and differentiable (!) function

$$F_t(Y) = \left\{\begin{array}{l} \min \quad \langle C, X\rangle + \frac{1}{2\,t}\|X - Y\|^2 \\ \quad \mathcal{A}X = b \\ \quad X \succcurlyeq 0 \end{array}\right.$$

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Hence

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Proximal point in practice

General fact:

$$\begin{array}{ll} & \text{gradient step on } F_t \\ \iff & \text{fixed point iteration on prox point} \\ \iff & Y_{k+1} = \operatorname{prox}(Y_k) \end{array}$$

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Practical implementation requires

- an algorithm to compute $\mathrm{prox}(Y_k) \longrightarrow$ SDLS
- a rule to stop this algorithm (in practice $Y_{k+1} \approx \mathrm{prox}(Y_k)$)

Solving the outer problem

Algorithm (General proximal algorithm for SDP)

Repeat until $\|Z + \mathcal{A}^* y - C\|$ *small:*

Repeat until $\|b - \mathcal{A}X\|$ *small enough:*

Compute $X = t(\mathcal{A}^* y - C + Y/t)_+$, $Z = -(\mathcal{A}^* y - C + Y/t)_-$

Set $g = b - \mathcal{A}X$

Update $y \leftarrow y + \tau W g$ *with appropriate* τ *and* W

end (inner repeat)

$Y \leftarrow X$

end (outer repeat)

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- Connexion with boundary points: If $\mathcal{A}$ is surjective,

$$W = [\mathcal{A}\mathcal{A}^*]^{-1} \qquad \tau = 1/t$$

  then: the present algorithm = the boundary point method

J. Malick, J. Povh, F. Rendl and A. Wiegele
Boundary point method to solve semidefinite programs
Computing, 2006

In practice

regularization algorithms:

$$\text{projections methods} \perp \text{interior points}$$

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After testing various prototypes, we chose: one single inner iteration, $W = [\mathcal{A}\mathcal{A}^*]^{-1}$, $\tau = 1/t$, update strategy of parameters...

Algorithm (Final algorithm)

Repeat until $\delta <$*tolerance:*
Compute the solution y *of* $\mathcal{A}\mathcal{A}^* y = \mathcal{A}(C - Z) + (b - \mathcal{A}Y)/t$
Compute $S = Y/t + \mathcal{A}^* y - C$, $Z = -S_-$ *and* $X = S_+$
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$\mathcal{A}\mathcal{A}^*$ (and its Choleski factorization) computed one single time !

Numerical experiments

$\longrightarrow$ 1st test: Random instances with large m

(Generator is online, so data reproducible)

`http//bipop.inrialpes.fr/people/malick/software`

$$(\text{SDP}) \left\{ \begin{array}{ll} \min & \langle C, X \rangle \\ & \mathcal{A}X = b \\ & X \succcurlyeq 0 \end{array} \right. \qquad (\text{dual}) \left\{ \begin{array}{ll} \max & b^\top y \\ & C - \mathcal{A}^* y = Z \\ & Z \succcurlyeq 0 \end{array} \right.$$

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⟶ Comparison with 2 strong codes - on a standard machine

K.C. Toh
Solving large scale semidefinite programs via an iterative solver on the augmented systems
SIAM Journal on Optimization, 2004.

S. Burer and R.D.C Monteiro
A nonlinear programming algorithm for solving SDP via low-rank factorization
Mathematical Programming, 2003

Random instances

Stopping tolerance for our regularization method:

$$\text{error} := \max\left\{\frac{\|\mathcal{A}X - b\|}{1+\|b\|}, \frac{\|C - Z - \mathcal{A}^*y\|}{1+\|C\|}\right\} \leqslant 10^{-7}$$

n	m	prox time
400	30000	153
500	30000	202
500	40000	217
700	50000	592
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n	m	time	time	error	time	error
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500	30000	202	1309	0.5e-7	2165	7.8e-6
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Note: If the factorization given, larger instances ($n = 1000$, $m = 150,000$) in less than one hour

Example in combinatorics: Lovász theta number

Let G be a graph (and $\overline{G}$ its complementary)

$$\textcolor{red}{\vartheta}(G) := \begin{cases} \max \quad \langle \mathbf{1}_{n\times n}, X\rangle \\ \quad X_{ij} = 0, \quad \forall\, i,j \in E(G) \\ \quad \operatorname{trace} X = 1 \\ \quad X \succcurlyeq 0 \end{cases}$$

It holds
$$\omega(G) \leqslant \vartheta(\overline{G}) \leqslant \chi(G)$$

The clique number $\omega(G)$ and the chromatic number $\chi(G)$ are "impossible" to compute and even hard to approximate

L. Lovász
On the Shannon capacity of a graph
IEEE Trans. Inform. Theory, 1979

Computing Lovász theta number

- Challenge: Lovász number of certain graphs of the DIMACS collection not computed so far...

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- Results: from DIMACS library

graph name	$\lvert V(G)\rvert$	$\lvert E(\overline{G})\rvert$	$\vartheta(\overline{G})$	$\lvert E(G)\rvert$	$\vartheta(G)$
brock400-1	400	20077	39.702	59723	10.388
keller5	776	74710	31.000	225990	31.000
brock800-1	800	112095	42.222	207505	19.233
p-hat500-1	500	93181	13.074	31569	58.036
p-hat1000-3	1000	127754	84.80	371746	18.23

Conclusion: regularization method

Our new method for solving SDP

$\longrightarrow$ follows a very general approach of convex analysis

$\longrightarrow$ takes an opposite direction to Interior Points

$\longrightarrow$ compares favorably with other solvers for some problems

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More work:

$\longrightarrow$ to understand the convergence ($\rightarrow$ tuning of parameters)

$\longrightarrow$ to enlarge the range of tackled problems

$\longrightarrow$ to handle inequalities and others cones

(Final) conclusion: SDLS and next

Semidefinite least-squares:
to project onto subsets of SDP cone

$$\text{(SDLS)} \begin{cases} \min & \|X - C\|^2 \\ & \mathcal{A}X = b \\ & X \succcurlyeq 0 \end{cases}$$

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⟶ Applications (still under development)

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2. bounding procedure in combinatorial optimization
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