A first example

A invertible $n \times n$ real matrix

Given $b \in \mathbb{R}^n$ find $x \in \mathbb{R}^n$ such that $Ax = b$.

$$A(x + \Delta x) = (b + \Delta b)$$

How large is Δx w.r.t. Δb?

$$\frac{||\Delta x||}{||x||} \leq \kappa(A) \frac{||\Delta b||}{||b||}$$

with

$$\kappa(A) = ||A|| ||A^{-1}||.$$

Errors in both A and b

$$\frac{\|\Delta x\|}{\|x\|} \leq \frac{\kappa(A)}{1-\kappa(A)\frac{\|\Delta A\|}{\|A\|}}\left(\frac{\|\Delta A\|}{\|A\|}+\frac{\|\Delta b\|}{\|b\|}\right).$$

Notice that for small $\|\Delta A\|$

$$\frac{\kappa(A)}{1-\kappa(A)\frac{\|\Delta A\|}{\|A\|}} \approx \kappa(A).$$

Conclusion: $\kappa(A)$ measures the sensitivity of the problem to small perturbations or, in other words, how well conditioned the input is with respect to the problem.

"We should describe the equations (8.2) as an *ill-conditioned* set, or, at any rate, as ill-conditioned when compared with (8.1). It is characteristic of ill-conditioned sets of equations that small percentage errors in the coefficients given may lead to large percentage errors in the solution."

Alan M. Turing [1948]

$\kappa(A)$ is the condition number of A.

If A is not invertible we let $\kappa(A) = \infty$.

$$A \text{ is said to be } \begin{cases} \text{well-conditioned} & \text{when } \kappa(A) \text{ is small} \\ \text{ill-conditioned} & \text{when } \kappa(A) \text{ is large} \\ \text{ill-posed} & \text{when } \kappa(A) = \infty \end{cases}$$

The Condition Number Theorem

Σ set of ill-posed matrices (measure zero in $\mathbb{R}^{n^2}$)

Theorem 5 (Eckart & Young, 1936) *For any $n \times n$ real matrix A one has*

$$\kappa(A) = \frac{\|A\|}{d_F(A, \Sigma)}.$$