Modélisation asymptotique de la diffraction d'ondes élastiques par des petits défauts débouchant à la surface.

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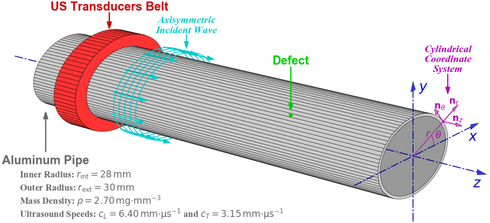
Motivation

Asymptotic approximations for (acoustic, elastic, EM) fields perturbed by small objects

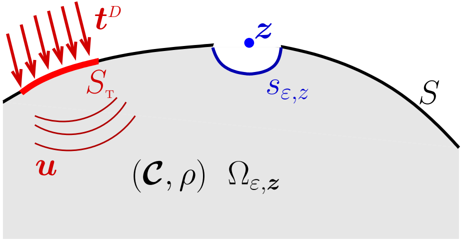
- Abundantly studied for objects **embedded** in (bounded or unbounded) media
  e.g. [Cedyo-Fengya, Moskow, Vogelius 99; Ammari, Kang 04; Claeys 08; Cassier, Hazard 12, Bendali, Cocquet, Tordeux 16 and **many many more**]
- Comparatively few studies on asymptotic models involving small **surface-breaking** defects (**SBDs**): indentations, emerging cracks, corrosion pits...
  e.g. [Dambrine, Vial 07] on 2D Laplace and elastostatics
  [Silva, Geubelle, Tortorelli 11] on 2D elastostatics with emerging cracks
- Such models potentially useful for e.g. moderate-cost simulations of NDT experiments involving SBDs;

Present motivation:

- Develop asymptotic models for ultrasound NDT on e.g. plates or tubes with small SBDs; Exploit availability at I2M of (semi-analytical) elastodynamic Green's tensors for such media;
  [PhD A. Krishna, I2M, 2020]
- Obtain mathematical results on asymptotic models for wave scattering by small SBDs;



Setting



(S smooth, traction-free in a neighborhood of $\boldsymbol{z}$)

Elastodynamic incident field $\boldsymbol{u}$:

$$\boldsymbol{\mathcal{L}}_\omega[\boldsymbol{u}] = \boldsymbol{0} \quad \text{in } \Omega \qquad (\text{with } \boldsymbol{\mathcal{L}}_\omega[\boldsymbol{w}] := -\text{div}\,[\boldsymbol{\mathcal{C}} : \boldsymbol{\nabla}^{\text{s}}\boldsymbol{w}] - \rho\omega^2\boldsymbol{w})$$

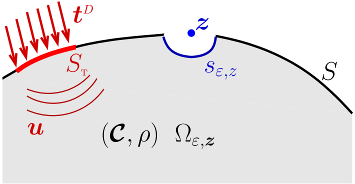
$$\boldsymbol{t}[\boldsymbol{u}] = \boldsymbol{t}^D \quad \text{on } S_{\text{T}} \qquad (\text{with } \boldsymbol{t}[\boldsymbol{w}] := \boldsymbol{n}\cdot\boldsymbol{\sigma}[\boldsymbol{w}] = \boldsymbol{n}\cdot\boldsymbol{\mathcal{C}} : \boldsymbol{\nabla}^{\text{s}}\boldsymbol{w})$$

$$\boldsymbol{t}[\boldsymbol{u}] = \boldsymbol{0} \quad \text{on } S \setminus S_{\text{T}}$$

Scattering of (given) incident field $\big(\boldsymbol{u}, \boldsymbol{t} := \boldsymbol{t}[\boldsymbol{u}]\big)$ by small indentation:

$$\boxed{\boldsymbol{u}_\varepsilon = \boldsymbol{u} + \boldsymbol{v}_\varepsilon} \quad \text{with} \quad \boldsymbol{\mathcal{L}}_\omega[\boldsymbol{v}_\varepsilon] = \boldsymbol{0} \ \text{ in } \Omega_{\varepsilon,z}, \quad \boldsymbol{t}[\boldsymbol{v}_\varepsilon] = -\boldsymbol{t}[\boldsymbol{u}] \ \text{ on } s_{\varepsilon,z}, \quad \boldsymbol{t}[\boldsymbol{v}_\varepsilon] = \boldsymbol{0} \ \text{ on } S \setminus s_{\varepsilon,z}$$

Governing integral equation



▷ Elastodymanic Green's tensor $\boldsymbol{G}_\omega = [\boldsymbol{G}^1_\omega \; \boldsymbol{G}^2_\omega \; \boldsymbol{G}^3_\omega]$

$$\boldsymbol{\mathcal{L}}_\omega \boldsymbol{G}^k_\omega(\cdot, \boldsymbol{x}) = \delta(\cdot - \boldsymbol{x})\boldsymbol{e}_k \text{ in } \Omega_\varepsilon, \qquad \boldsymbol{t}\big[\boldsymbol{G}^k_\omega(\cdot, \boldsymbol{x})\big] = \boldsymbol{0} \text{ on } S \qquad (k = 1, 2, 3)$$

▷ Governing boundary integral equation (BIE) for the scattered field $\boldsymbol{v}_\varepsilon$:

$$\Big(\frac{1}{2}\boldsymbol{I} + \mathcal{H}_\varepsilon\Big)\boldsymbol{v}_\varepsilon = -\mathcal{G}_\varepsilon \boldsymbol{t} \qquad \begin{cases} \mathcal{H}_\varepsilon \boldsymbol{v}(\boldsymbol{x}) := \text{p.v.} \displaystyle\int_{s_{\varepsilon,z}} \boldsymbol{v}(\boldsymbol{y}) \cdot \boldsymbol{t}[\boldsymbol{G}_\omega(\boldsymbol{y}, \boldsymbol{x})] \, \mathrm{d}\boldsymbol{y}, \\ \mathcal{G}_\varepsilon \boldsymbol{t}(\boldsymbol{x}) := \displaystyle\int_{s_{\varepsilon,z}} \boldsymbol{t}(\boldsymbol{y}) \cdot \boldsymbol{G}_\omega(\boldsymbol{y}, \boldsymbol{x}) \, \mathrm{d}\boldsymbol{y}, \end{cases} \qquad (\boldsymbol{x} \in s_{\varepsilon,z})$$

- Other (e.g. Galerkin) BIE formulations possible
- BIE framework useful for problem formulation / analysis
- Choice of actual computational method remains open

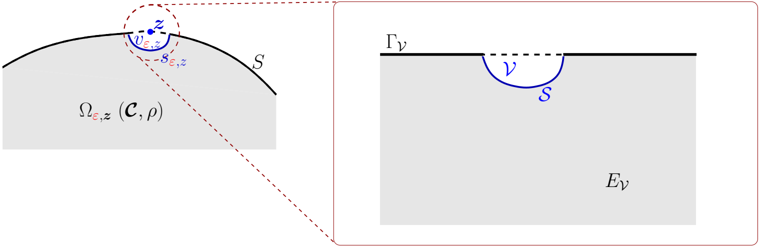
Governing integral equation

Governing boundary integral equation (BIE) for the scattered field $v_{\varepsilon,z}$:

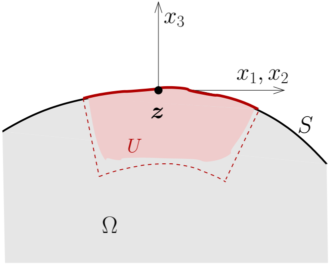
$$\Big(\frac{1}{2}\boldsymbol{I}+\mathcal{H}_{\varepsilon}\Big)\boldsymbol{v}_{\varepsilon}=-\mathcal{G}_{\varepsilon}\boldsymbol{t}$$

Goal:

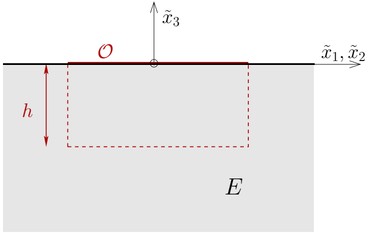
- ▷ Find leading-order approximation of $\boldsymbol{v}_{\varepsilon}$ as $\varepsilon\to 0$;
- ▷ Seek limiting form of BIE.
  Expectation: elastostatic problem in half-space with normalized indentation



Local parametrization using rectified coordinates



$\boldsymbol{x} \in \Omega$ (physical coordinates)



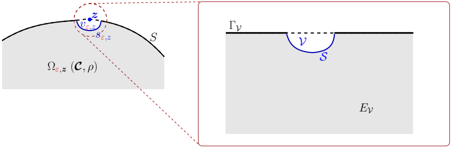
$\widetilde{\boldsymbol{x}} \in E$ (rectified coordinates)

▷ Surface S: $\boldsymbol{x} = \boldsymbol{z} + \begin{Bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ F(\tilde{x}_1, \tilde{x}_2) \end{Bmatrix}, \qquad (\tilde{x}_1, \tilde{x}_2) \in \mathcal{O}$

▷ Domain Ω: $\boldsymbol{x} = \boldsymbol{z} + \begin{Bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ F(\tilde{x}_1, \tilde{x}_2) \end{Bmatrix} + \tilde{x}_3 \boldsymbol{n}(\tilde{x}_1, \tilde{x}_2) =: \boldsymbol{\Phi}(\widetilde{\boldsymbol{x}}), \qquad \widetilde{\boldsymbol{x}} \in \mathcal{U} := \mathcal{O} \times [-h, 0]$

▷ Additional assumptions (WLOG): $F = \partial_1 F = \partial_2 F = \partial_{12} F = 0$ at $(\tilde{x}_1, \tilde{x}_2) = (0, 0)$.
$\Longrightarrow \ \boldsymbol{n}(0,0) = \boldsymbol{e}_3, \ \boldsymbol{\Phi}(\boldsymbol{0}) = \boldsymbol{z}, \ \boldsymbol{\nabla\Phi}(\boldsymbol{0}) = \boldsymbol{I}.$

Indentations of vanishing size



- Choose shape $\mathcal{V}$ of limiting indentation (e.g. $\mathcal{V}$ is half the unit sphere);
- Define family of indentations: $\boldsymbol{x} = \boldsymbol{\Phi}(\varepsilon\bar{\boldsymbol{x}}),\ \bar{\boldsymbol{x}} \in \mathcal{V}$, i.e. $v_{\varepsilon,z} = \boldsymbol{\Phi}(\varepsilon\mathcal{V})$
- We have $\boldsymbol{x} = \boldsymbol{z} + \varepsilon\bar{\boldsymbol{x}} + O(\varepsilon^2)$ (since $\boldsymbol{\Phi}(\boldsymbol{0}) = \boldsymbol{z}$, $\boldsymbol{\nabla\Phi}(\boldsymbol{0}) = \boldsymbol{I}$)
  By contrast to embedded-cavity case, $v_{\varepsilon,z}$ asymptotically self-similar only.
- Let $\mathcal{S} := E \cap \partial\mathcal{V}$, then:

$$\boldsymbol{n}(\boldsymbol{x}) = \bar{\boldsymbol{n}}(\bar{\boldsymbol{x}}) + O(\varepsilon), \quad \mathrm{d}S(\boldsymbol{x}) = \varepsilon^2\big(1 + O(\varepsilon)\big)\,\mathrm{d}S(\bar{\boldsymbol{x}}) \qquad (\boldsymbol{x} \in s_{\varepsilon,z} = \boldsymbol{\Phi}(\varepsilon\mathcal{S}))$$

Recall governing BIE

$$\frac{1}{2}\boldsymbol{v}_\varepsilon(\boldsymbol{x}) + \int_{s_{\varepsilon,z}} \boldsymbol{v}_\varepsilon(\boldsymbol{y})\cdot\boldsymbol{t}[\boldsymbol{G}_\omega(\boldsymbol{y},\boldsymbol{x})]\,\mathrm{d}\boldsymbol{y} = -\int_{s_{\varepsilon,z}} \boldsymbol{t}[\boldsymbol{u}](\boldsymbol{y})\cdot\boldsymbol{G}_\omega(\boldsymbol{y},\boldsymbol{x})\,\mathrm{d}\boldsymbol{y} \quad \boldsymbol{x} \in s_{\varepsilon,z}$$

(a) set $\boldsymbol{y} = \boldsymbol{\Phi}(\varepsilon\bar{\boldsymbol{y}})$ and $\boldsymbol{x} = \boldsymbol{\Phi}(\varepsilon\bar{\boldsymbol{x}})$ (BIE with fixed support $\mathcal{S}$);

(b) seek resulting limiting form

(c) Interpret (b) in terms of a BVP

Main tool for (b): suitable representation of $\boldsymbol{G}_\omega$.

Additive decomposition of Green's tensor

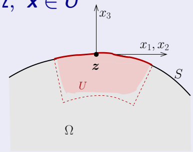
Let $(\widetilde{\boldsymbol{y}},\widetilde{\boldsymbol{x}})\in E\times E\mapsto \boldsymbol{G}_0^E(\widetilde{\boldsymbol{y}},\widetilde{\boldsymbol{x}})$: elastostatic Green's tensor with traction-free BC on ∂E.

- ▷ Homogeneity (important!): $\boldsymbol{G}_0^E(\lambda\widetilde{\boldsymbol{y}},\lambda\widetilde{\boldsymbol{x}})=\lambda^{-1}\boldsymbol{G}_0^E(\widetilde{\boldsymbol{y}},\widetilde{\boldsymbol{x}})$ $(\lambda>0)$

Additive decomposition of elastostatic Green's tensor

There exist tensor-valued functions $\boldsymbol{K},\boldsymbol{H}_0$ such that

$$\boldsymbol{G}_0(\boldsymbol{y},\boldsymbol{x})=\chi_U(\boldsymbol{y})\big(\boldsymbol{G}_0^E(\widetilde{\boldsymbol{y}},\widetilde{\boldsymbol{x}})+\widetilde{\boldsymbol{x}}\cdot\boldsymbol{K}(\widetilde{\boldsymbol{y}},\widetilde{\boldsymbol{x}};\widetilde{\boldsymbol{x}})\big)+\boldsymbol{H}_0(\boldsymbol{y},\boldsymbol{x})\qquad \boldsymbol{y}\in\Omega,\ \boldsymbol{x}\in U$$



where χ_U: C^∞ cutoff function ($\chi=1$ in U) and

(a) $(\boldsymbol{a},\boldsymbol{b})\mapsto\boldsymbol{K}(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c})$ is positively homogeneous with degree -1,

(b) $\boldsymbol{c}\mapsto\boldsymbol{K}(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c})$ is C^0 for all $\boldsymbol{a}\neq\boldsymbol{b}$,

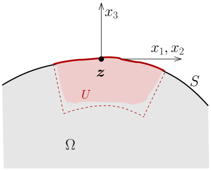
(c) $\boldsymbol{H}_0(\cdot,\boldsymbol{x})\in H^1(U)$ with $\|\boldsymbol{H}_0(\cdot,\boldsymbol{x})\|_{H^1(U)}\leq C$ uniformly in $\boldsymbol{x}\in U$.

Proof outline:

- ▷ Evaluate $\boldsymbol{\mathcal{L}}_0[\boldsymbol{G}_0^E(\widetilde{\boldsymbol{y}},\widetilde{\boldsymbol{x}})]$ ($\boldsymbol{\mathcal{L}}_0$ acting on $\boldsymbol{y}$)
- ▷ Define homogeneous singular correction $\boldsymbol{K}$ s.t. $\boldsymbol{\mathcal{L}}_0[\boldsymbol{G}_0^E(\widetilde{\boldsymbol{y}},\widetilde{\boldsymbol{x}})+\widetilde{\boldsymbol{x}}\cdot\boldsymbol{K}(\widetilde{\boldsymbol{y}},\widetilde{\boldsymbol{x}};\widetilde{\boldsymbol{x}})]=\delta\boldsymbol{I}+\ldots$
  ($\boldsymbol{K}$ governed by PD operator in $\widetilde{\boldsymbol{y}}$ with coefs involving $\boldsymbol{\Phi}(\widetilde{\boldsymbol{x}})$)
- ▷ Find governing BVP for nonsingular correction $\boldsymbol{H}_0$ by superposition

Green's tensor asymptotics

$$\boldsymbol{G}_\omega(\boldsymbol{y},\boldsymbol{x}) = \chi_U(\boldsymbol{y})\big(\boldsymbol{G}_0^E(\widetilde{\boldsymbol{y}},\widetilde{\boldsymbol{x}}) + \widetilde{\boldsymbol{x}}\cdot\boldsymbol{K}(\widetilde{\boldsymbol{y}},\widetilde{\boldsymbol{x}};\widetilde{\boldsymbol{x}})\big) + \boldsymbol{H}(\boldsymbol{y},\boldsymbol{x})$$



Interpretation of terms:

- $\boldsymbol{G}_0^E(\widetilde{\boldsymbol{y}},\widetilde{\boldsymbol{x}})$: main singular part of $\boldsymbol{G}_\omega$;
- $\boldsymbol{K}(\widetilde{\boldsymbol{y}},\widetilde{\boldsymbol{x}};\widetilde{\boldsymbol{x}})$: singular (and homogeneous) correction induced by curvature of S; $\boldsymbol{K}(\widetilde{\boldsymbol{y}},\widetilde{\boldsymbol{x}};\widetilde{\boldsymbol{x}}) = \boldsymbol{0}$ if S flat in U
- $\boldsymbol{H}(\widetilde{\boldsymbol{y}},\widetilde{\boldsymbol{x}};\widetilde{\boldsymbol{x}})$: nonsingular complementary term. includes $\boldsymbol{G}_\omega - \boldsymbol{G}_0$

Evaluation at $\boldsymbol{y},\boldsymbol{x} \in v_{\varepsilon,z}$, i.e. $(\boldsymbol{y},\boldsymbol{x}) = \boldsymbol{\Phi}(\varepsilon\bar{\boldsymbol{y}},\varepsilon\bar{\boldsymbol{x}})$, $(\widetilde{\boldsymbol{y}},\widetilde{\boldsymbol{x}}) = \varepsilon(\bar{\boldsymbol{x}},\bar{\boldsymbol{y}})$:

$$\boldsymbol{G}_\omega(\boldsymbol{y},\boldsymbol{x}) = \varepsilon^{-1}\boldsymbol{G}_0^E(\bar{\boldsymbol{y}},\bar{\boldsymbol{x}}) + \boldsymbol{R}_\varepsilon(\bar{\boldsymbol{y}},\bar{\boldsymbol{x}})$$

$$\text{with}\quad \boldsymbol{R}_\varepsilon(\bar{\boldsymbol{y}},\bar{\boldsymbol{x}}) := \bar{\boldsymbol{x}}\cdot\boldsymbol{K}(\bar{\boldsymbol{y}},\bar{\boldsymbol{x}};\varepsilon\bar{\boldsymbol{x}}) + \boldsymbol{H}(\varepsilon\bar{\boldsymbol{y}},\varepsilon\bar{\boldsymbol{x}}) = O(1)$$

$$\boldsymbol{t}[\boldsymbol{G}_\omega^k(\boldsymbol{y},\boldsymbol{x})] = \varepsilon^{-2}\boldsymbol{t}[\boldsymbol{G}_0^E(\bar{\boldsymbol{y}},\bar{\boldsymbol{x}})] + \varepsilon^{-1}\boldsymbol{t}[\boldsymbol{R}_\varepsilon(\bar{\boldsymbol{y}},\bar{\boldsymbol{x}})]$$

$$\text{with}\quad \boldsymbol{t}[\boldsymbol{R}_\varepsilon(\bar{\boldsymbol{y}},\bar{\boldsymbol{x}})] := \boldsymbol{t}\big[\bar{\boldsymbol{x}}\cdot\boldsymbol{K}(\bar{\boldsymbol{y}},\bar{\boldsymbol{x}};\varepsilon\bar{\boldsymbol{x}})\big] + \varepsilon\,\boldsymbol{t}[\boldsymbol{H}(\varepsilon\bar{\boldsymbol{y}},\varepsilon\bar{\boldsymbol{x}})] = O(1)$$

Limiting form of governing integral equation

$$\boldsymbol{G}_\omega(\boldsymbol{y},\boldsymbol{x}) = \varepsilon^{-1}\big\{\,\boldsymbol{G}_0^E(\bar{\boldsymbol{y}},\bar{\boldsymbol{x}})+O(\varepsilon)\,\big\}, \qquad \boldsymbol{t}[\boldsymbol{G}_\omega^k(\boldsymbol{y},\boldsymbol{x})] = \varepsilon^{-2}\big\{\,\boldsymbol{t}[\boldsymbol{G}_0^E(\bar{\boldsymbol{y}},\bar{\boldsymbol{x}})]+O(\varepsilon)\,\big\} \qquad (\star)$$

Governing BIE:

$$\frac{1}{2}\boldsymbol{v}_\varepsilon(\boldsymbol{x}) + \text{p.v.}\int_{s_{\varepsilon,z}} \boldsymbol{v}_\varepsilon(\boldsymbol{y})\cdot\boldsymbol{t}[\boldsymbol{G}_\omega](\boldsymbol{y},\boldsymbol{x})\,\mathrm{d}\boldsymbol{y} = -\int_{s_{\varepsilon,z}} \boldsymbol{t}[\boldsymbol{u}](\boldsymbol{y})\cdot\boldsymbol{G}_\omega(\boldsymbol{y},\boldsymbol{x})\,\mathrm{d}\boldsymbol{y} \qquad \boldsymbol{x}\in s_{\varepsilon,z}$$

▷ Set $(\boldsymbol{y},\boldsymbol{x}) = \boldsymbol{\Phi}(\varepsilon\bar{\boldsymbol{y}},\varepsilon\bar{\boldsymbol{x}})$, $\mathrm{d}\boldsymbol{y} = (1+O(\varepsilon))\varepsilon^2\,\mathrm{d}\bar{\boldsymbol{y}}$,
use $(\star)$, observe $\boldsymbol{t}[\boldsymbol{u}](\boldsymbol{y}) = \boldsymbol{n}(\bar{\boldsymbol{y}})\cdot\boldsymbol{\sigma}(\boldsymbol{z})+O(\varepsilon)$,
define $\overline{\boldsymbol{v}}_\varepsilon(\bar{\boldsymbol{x}}) := \boldsymbol{v}_\varepsilon\big(\boldsymbol{\Phi}(\varepsilon\bar{\boldsymbol{x}})\big)$: $\Longrightarrow$

limiting form (i.e. leading-order part) of BIE:

$$\boxed{\Big(\frac{1}{2}\boldsymbol{I}+\mathcal{H}_E\Big)\overline{\boldsymbol{v}}_\varepsilon = -\varepsilon\mathcal{G}_E\big(\boldsymbol{n}\cdot\boldsymbol{\sigma}[\boldsymbol{u}](\boldsymbol{z})\big) + o(\varepsilon)}$$

$$\begin{cases} \mathcal{H}_E\boldsymbol{v}(\bar{\boldsymbol{x}}) := \text{p.v.}\displaystyle\int_{\mathcal{S}} \boldsymbol{v}(\bar{\boldsymbol{y}})\cdot\boldsymbol{t}[\boldsymbol{G}_0^E(\bar{\boldsymbol{y}},\bar{\boldsymbol{x}})]\,\mathrm{d}\bar{\boldsymbol{y}}, \\ \mathcal{G}_E\boldsymbol{t}(\bar{\boldsymbol{x}}) := \displaystyle\int_{\mathcal{S}} \boldsymbol{v}(\bar{\boldsymbol{y}})\cdot\boldsymbol{G}_0^E(\bar{\boldsymbol{y}},\bar{\boldsymbol{x}})\,\mathrm{d}\bar{\boldsymbol{y}}, \end{cases} \qquad (\bar{\boldsymbol{x}}\in\mathcal{S})$$

Resulting ansatz: $\overline{\boldsymbol{v}}_\varepsilon(\bar{\boldsymbol{y}}) = \varepsilon\,\boldsymbol{V}(\bar{\boldsymbol{y}})+o(\varepsilon),\ \bar{\boldsymbol{y}}\in\mathcal{S}$

Limiting form of scattered field on indentation surface $\mathcal{S}$

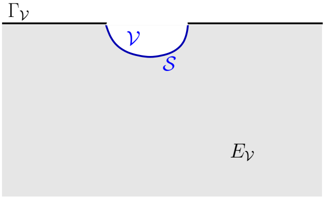
$$\boldsymbol{v}_\varepsilon(\boldsymbol{y}) = \varepsilon \boldsymbol{V}(\bar{\boldsymbol{y}}) + \boldsymbol{\delta}_\varepsilon(\boldsymbol{y}), \qquad \bar{\boldsymbol{y}} \in \mathcal{S},\ \boldsymbol{y} = \boldsymbol{\Phi}(\varepsilon \bar{\boldsymbol{y}})$$

where $\boldsymbol{V} \in \boldsymbol{H}^{1/2}(\mathcal{S})$ solves the normalized BIE

$$\Big(\frac{1}{2}\boldsymbol{I} + \mathcal{H}_E \Big)\boldsymbol{V} = -\mathcal{G}_E\big(\boldsymbol{n}\cdot\boldsymbol{\sigma}[\boldsymbol{u}](\boldsymbol{z})$$

governing the elastostatic BVP

$$\boldsymbol{\mathcal{L}}_0[\boldsymbol{V}] = \boldsymbol{0} \ \text{ in } E_\mathcal{V}, \qquad \boldsymbol{t}[\boldsymbol{V}] = \boldsymbol{0} \ \text{ on } \Gamma_\mathcal{V}, \quad \boldsymbol{t}[\boldsymbol{V}] = -\boldsymbol{n}\cdot\boldsymbol{\sigma}[\boldsymbol{u}](\boldsymbol{z}) \ \text{ on } \mathcal{S}, \quad |\boldsymbol{V}| \to 0 \ \text{ at } \infty$$



By linear superposition: $\boldsymbol{V} = \sigma_{ij}(\boldsymbol{z})\boldsymbol{W}^{ij} \quad 1 \leq i,j \leq 2 \qquad$ (since $\boldsymbol{\sigma}_{i3}(\boldsymbol{z}) = 0$) with

$$\boldsymbol{\mathcal{L}}_0[\boldsymbol{W}^{ij}] = \boldsymbol{0} \ \text{ in } E_\mathcal{V},$$
$$\boldsymbol{t}[\boldsymbol{W}^{ij}] = \boldsymbol{0} \ \text{ on } \Gamma_\mathcal{V}, \quad \boldsymbol{t}[\boldsymbol{W}^{ij}] = -\tfrac{1}{2}\big(n_i\boldsymbol{e}_j + n_j\boldsymbol{e}_i\big) \ \text{ on } \mathcal{S}, \quad |\boldsymbol{W}^{ij}| \to 0 \ \text{ at } \infty$$

Asymptotic approximation of scattered field

- Integral representation:

$$\boldsymbol{v}_\varepsilon(\boldsymbol{x}) = -\int_{S_{\varepsilon,z}} \boldsymbol{v}_\varepsilon \cdot \boldsymbol{t}[\boldsymbol{G}_\omega](\cdot,\boldsymbol{x})\,\mathrm{d}S + \int_{V_{\varepsilon,z}} \big[\boldsymbol{\sigma} : \boldsymbol{\nabla}^{\mathrm{s}}\boldsymbol{G}_\omega(\cdot,\boldsymbol{x}) - \rho\omega^2\boldsymbol{u}\cdot\boldsymbol{G}_\omega(\cdot,\boldsymbol{x})\big]\,\mathrm{d}V \qquad \boldsymbol{x}\in\Omega_{\varepsilon,z}$$

- Set $(\boldsymbol{y},\boldsymbol{x}) = \boldsymbol{\Phi}(\varepsilon\bar{\boldsymbol{y}},\varepsilon\bar{\boldsymbol{x}}),\ \mathrm{d}S = (1+O(\varepsilon))\varepsilon^2\,\mathrm{d}\overline{S},\ \mathrm{d}V = (1+O(\varepsilon))\varepsilon^3\,\mathrm{d}\overline{V}$
- use expansions $\boldsymbol{v}_\varepsilon = \varepsilon\sigma_{ij}(\boldsymbol{z})\boldsymbol{W}^{ij}(\bar{\boldsymbol{y}}) + o(\varepsilon), \qquad \boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{z}) + o(1),$
  $\boldsymbol{G}_\omega(\cdot,\boldsymbol{x}) = \boldsymbol{G}_\omega(\boldsymbol{z},\boldsymbol{x}) + o(1), \qquad \boldsymbol{\nabla}^{\mathrm{s}}\boldsymbol{G}_\omega(\cdot,\boldsymbol{x}) = \boldsymbol{\nabla}^{\mathrm{s}}\boldsymbol{G}_\omega(\boldsymbol{z},\boldsymbol{x}) + o(1)$

Asymptotic approximation of scattered field

$$\boldsymbol{v}_\varepsilon(\boldsymbol{x}) = \varepsilon^3|\mathcal{V}|\big[\boldsymbol{\sigma}[\boldsymbol{G}_\omega](\boldsymbol{z},\boldsymbol{x}) : \boldsymbol{\mathcal{A}} : \boldsymbol{\sigma}(\boldsymbol{z}) - \rho\omega^2\boldsymbol{G}_\omega(\boldsymbol{z},\boldsymbol{x})\cdot\boldsymbol{u}(\boldsymbol{z})\big] + o(\varepsilon^3),$$

where $\boldsymbol{\mathcal{A}}$ **elastic moment tensor** (EMT):

$$\boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{C}}^{-1} - \frac{1}{|\mathcal{V}|}\Big\{\int_{\mathcal{S}} \boldsymbol{n}\otimes\boldsymbol{W}\,\mathrm{d}S\Big\}.$$

Remarks on found asymptotic representation of $\boldsymbol{v}_\varepsilon$

Notable properties of EMT:

- $\boldsymbol{\mathcal{A}}$ depends on $\boldsymbol{n}(\boldsymbol{z})$, otherwise independent of local geometry of Ω at $\boldsymbol{z}$;
- $\boldsymbol{\mathcal{A}}$ has same (major, minor) symmetries as $\boldsymbol{\mathcal{C}}$;
- Free surface BC implies $\sigma_{i3}(\boldsymbol{z}) = 0$ and $[\Sigma_\omega]_{i3}(\boldsymbol{z},\boldsymbol{x}) = 0$, so $\boldsymbol{\mathcal{A}}_{ijk\ell}$ only nonzero entries of $\boldsymbol{\mathcal{A}}$;
- $\boldsymbol{\mathcal{A}}$ depends on $\mathcal{S}$ and $\boldsymbol{\mathcal{C}}$ only, hence same EMT applies (up to rotations) for any Ω and site $\boldsymbol{z}$ once $\mathcal{S}$ chosen.

Notable characteristics of asymptotic representation of $\boldsymbol{v}_\varepsilon$:

- ▷ structure of formula identical to that for an embedded defect;
- ▷ depends through $\boldsymbol{\mathcal{A}}$ on $\boldsymbol{\mathcal{C}}$ and indentation limiting shape $\mathcal{S}$;
- ▷ depends through $\boldsymbol{G}_\omega$ on ω, domain geometry Ω, and material $\boldsymbol{\mathcal{C}}, \rho$;
- ▷ depends on curvature of S through $\boldsymbol{G}_\omega$ only;
- ▷ approximation of $\boldsymbol{v}_\varepsilon(\boldsymbol{x})$ combines monopolar and dipolar sources at $\boldsymbol{z} \in S$ with $O(\varepsilon^3|\mathcal{V}|)$ strength.
- ▷ no requirement (beyond C^2 smoothness) on local geometry of S near $\boldsymbol{z}$ (e.g. local convexity not required)

Justification of asymptotic expansion (under construction!)

Recall $\boldsymbol{G}_\omega(\boldsymbol{y},\boldsymbol{x}) = \varepsilon^{-1}\boldsymbol{G}_0^E(\bar{\boldsymbol{y}},\bar{\boldsymbol{x}}) + \boldsymbol{R}_\varepsilon(\bar{\boldsymbol{y}},\bar{\boldsymbol{x}})$, $\boldsymbol{t}[\boldsymbol{G}_\omega^k(\boldsymbol{y},\boldsymbol{x})] = \varepsilon^{-2}\boldsymbol{t}[\boldsymbol{G}_0^E(\bar{\boldsymbol{y}},\bar{\boldsymbol{x}})] + \varepsilon^{-1}\boldsymbol{t}[\boldsymbol{R}_\varepsilon(\bar{\boldsymbol{y}},\bar{\boldsymbol{x}})]$.

- Integral equations (in terms of $\boldsymbol{H}^{1/2}(\mathcal{S}) \to \boldsymbol{H}^{1/2}(\mathcal{S})$ integral operators):

$$\begin{aligned} &\mathcal{H}_E[\overline{\boldsymbol{v}}_\varepsilon](\bar{\boldsymbol{x}}) + \varepsilon\mathcal{Q}_\varepsilon[\overline{\boldsymbol{v}}_\varepsilon](\bar{\boldsymbol{x}}) = \varepsilon\mathcal{G}_E\big[\boldsymbol{n}\cdot\boldsymbol{\sigma}(\boldsymbol{\Phi}(\varepsilon\cdot))\big](\bar{\boldsymbol{x}}) + \varepsilon^2\mathcal{R}_\varepsilon\big[\boldsymbol{n}\cdot\boldsymbol{\sigma}(\boldsymbol{\Phi}(\varepsilon\cdot))\big](\bar{\boldsymbol{x}}) \\ &\mathcal{H}_E[\varepsilon V](\bar{\boldsymbol{x}}) = \varepsilon\mathcal{G}_E\big[\boldsymbol{n}\cdot\boldsymbol{\sigma}(\boldsymbol{z})\big](\bar{\boldsymbol{x}}) \end{aligned}$$

- Truncation error $\overline{\boldsymbol{\delta}}_\varepsilon := \overline{\boldsymbol{v}}_\varepsilon - \varepsilon\boldsymbol{V}$ solves

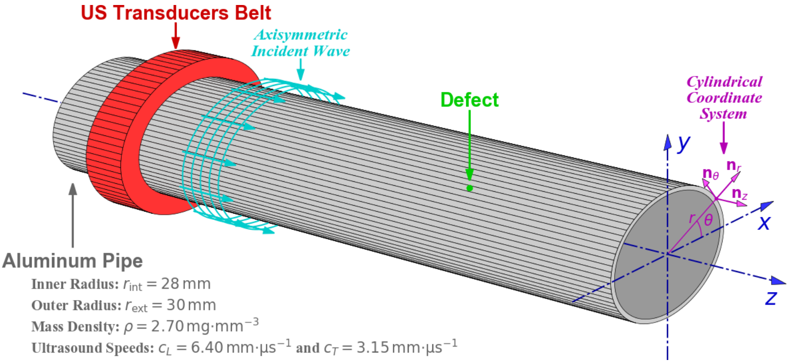
$$(\mathcal{H}_E + \mathcal{N}_\varepsilon)[\overline{\boldsymbol{\delta}}_\varepsilon](\bar{\boldsymbol{x}}) = \mathcal{F}_\varepsilon(\bar{\boldsymbol{x}})$$

with $\mathcal{F}_\varepsilon(\bar{\boldsymbol{x}}) := \varepsilon\mathcal{G}_E\big[\boldsymbol{n}\cdot\boldsymbol{\sigma}(\boldsymbol{\Phi}(\varepsilon\cdot)) - \boldsymbol{n}\cdot\boldsymbol{\sigma}(\boldsymbol{z})\big](\bar{\boldsymbol{x}}) + \varepsilon^2\mathcal{R}_\varepsilon\big[\boldsymbol{n}\cdot\boldsymbol{\sigma}(\boldsymbol{\Phi}(\varepsilon\cdot))\big](\bar{\boldsymbol{x}}) - \varepsilon^2\mathcal{Q}_\varepsilon[\boldsymbol{V}](\bar{\boldsymbol{x}})$

- Method of proof (ongoing):
  (a) $\mathcal{H}_E + \varepsilon\mathcal{Q}_\varepsilon = \mathcal{H}_E\big(\mathcal{I} + \varepsilon\mathcal{H}_E^{-1}\mathcal{N}_\varepsilon\big)$: $\boldsymbol{H}^{1/2}(\mathcal{S}) \to \boldsymbol{H}^{1/2}(\mathcal{S})$ boundedly invertible, uniformly in ε for small enough ε;
  (b) $\|\mathcal{F}_\varepsilon\|_{\boldsymbol{H}^{1/2}(\mathcal{S})} \leq C\varepsilon^2$
  (c) Hence $\|\overline{\boldsymbol{\delta}}_\varepsilon\|_{\boldsymbol{H}^{1/2}(\mathcal{S})} \leq C\varepsilon\|\varepsilon\boldsymbol{V}\|_{\boldsymbol{H}^{1/2}(\mathcal{S})}$
  (d) Invoke behavior of $\boldsymbol{H}^{1/2}$ norm under scaling

- Results in $\|\boldsymbol{\delta}_\varepsilon\|_{\boldsymbol{H}^{1/2}(v_{\varepsilon,z})} \leq C\varepsilon^{1/2}\|\boldsymbol{v}_\varepsilon\|_{\boldsymbol{H}^{1/2}(v_{\varepsilon,z})}$

Example



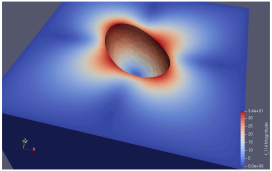
- Asymptotic model implemented for tubular geometries
- Uses existing in-house (I2M Bordeaux) implementation of (semi-analytic) elastodynamic Green's tensor

(plates) P. Mora, E. Ducasse, M.Deschamps (2016), Transient 3D elastodynamic field in an embedded multilayered anisotropic plate, *Ultrasonics* 69:106-115

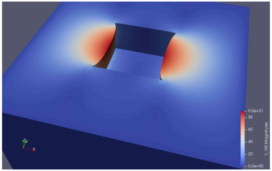
(pipes) A. Krishna, Topological imaging of tubular structures using ultrasonic guided waves. PhD Thesis (defense expected fall 2019)

Computation of EMT

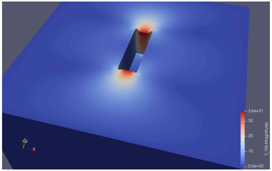
- ▷ FEM computation (Fenics) on truncated half-space (truncation dist. $\approx 5\times$indent. radius
- ▷ Some meshing issues (local refinement near indentation) still partially unresolved, so EMT accurate to about 3 digits only
- ▷ EMT for several indentation shapes:



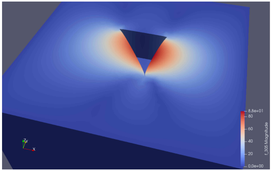
half-sphere



half-cube

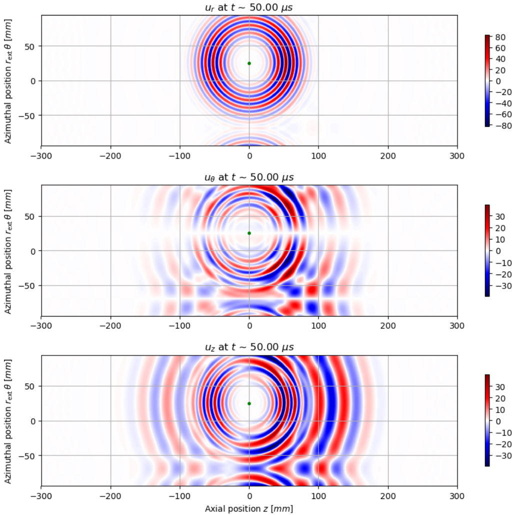


half-parallelipiped $4\times 1\times 2$



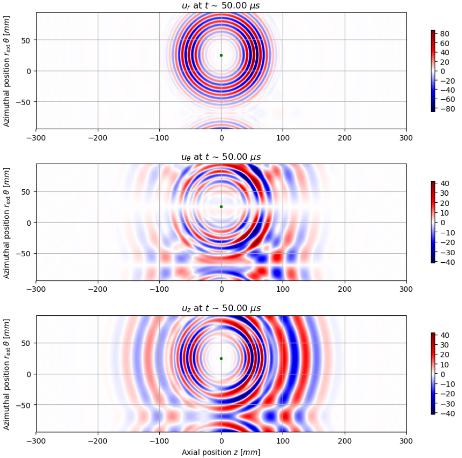
half-prism

Asymptotic model: scattering by half-spherical indentation, longitudinal (axisymmetric) incident mode

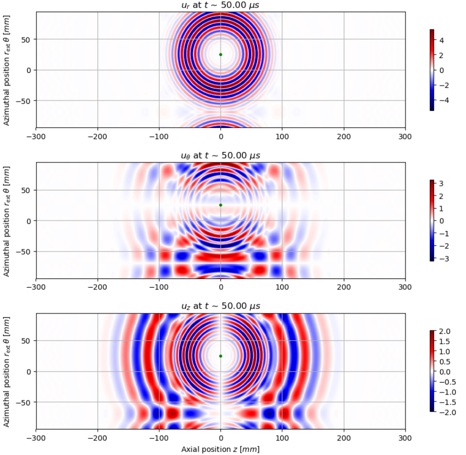


scattered

Asymptotic model: scattering by half-cubic indentation, longitudinal (axisymmetric) incident mode

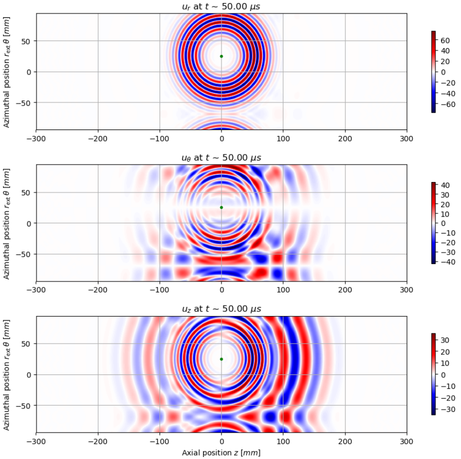


scattered

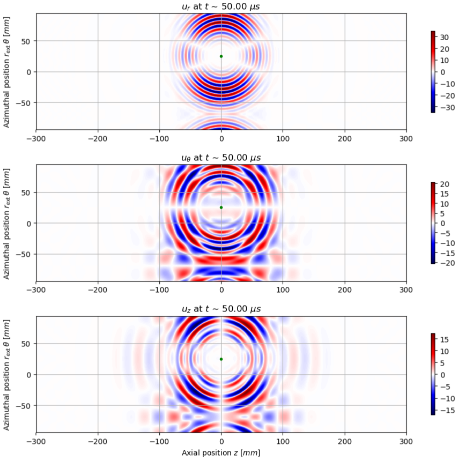


diff. w.r.t. half-sphere

Asymptotic model: scattering by half-parallelipiped indentation, longitudinal (axisymmetric) incident mode

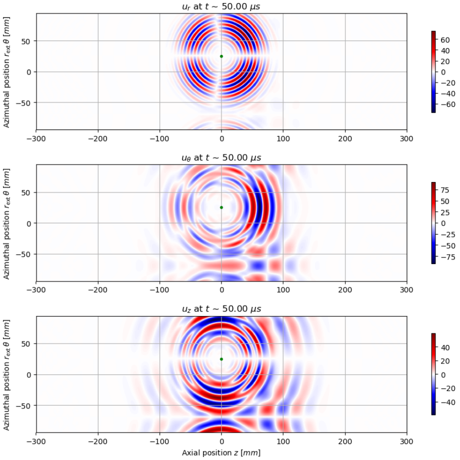


scattered

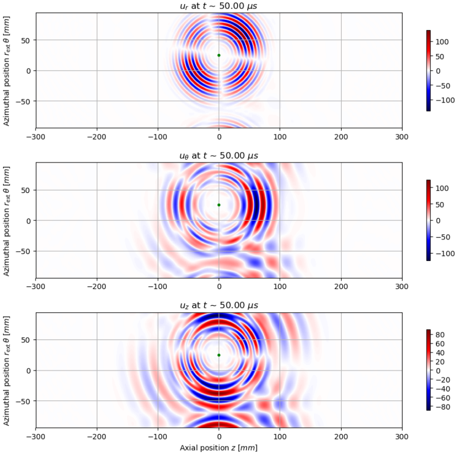


diff. w.r.t. half-sphere

Asymptotic model: scattering by half-sphere or prism indentation, torsional incident mode



scattered (half-sphere)

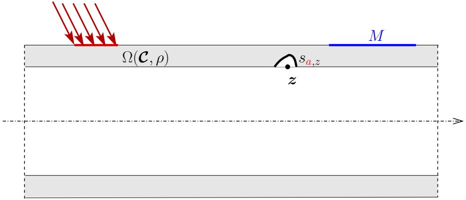


scattered (prism)

Concept of topological derivative

- Objective functional ($\boldsymbol{u}_{\mathrm{V}}$: total field due to surface defect (void) V):

$$\mathcal{J}(V) = J(\boldsymbol{u}_{\mathrm{V}}), \quad \text{e.g. (inversion):} \quad J(\boldsymbol{u}_{\mathrm{V}}) = \frac{1}{2}\int_M |\boldsymbol{u}_{\mathrm{V}} - \boldsymbol{u}_{\mathrm{obs}}|^2 \, \mathrm{d}M \quad \text{(output least squares)}$$



- Consider small indentation $v_{\varepsilon,z} = \boldsymbol{\Phi}(\varepsilon\mathcal{V})$ with surface $s_{\varepsilon,z}$

Definition (topological derivative)

Assume $\eta(\varepsilon) > 0$ *with* $\lim_{\varepsilon\to 0} \eta(\varepsilon) = 0$ *exists such that*

$$J(\boldsymbol{u}_\varepsilon) = J(\boldsymbol{u}) + \eta(\varepsilon)\mathcal{T}(\boldsymbol{z};\mathcal{B}) + o(\eta(\varepsilon)) \qquad (\boldsymbol{u}\text{: background displacement field})$$

Then $\mathcal{T}(\boldsymbol{z};\mathcal{B})$ *called* **topological derivative** *(TD) of* J *at* $\boldsymbol{z} \in \Omega$*.*

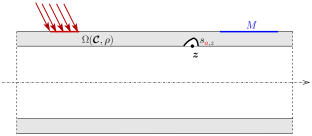
- TD: sensitivity analysis tool [Sokolowski, Zochowski 99; Garreau et al 01...] initially proposed for topology optimization [Eschenauer et al 94; Allaire et al 05...]

Formulation of TD using adjoint solution

Cost functional expansion: $$J(\boldsymbol{u}_\varepsilon) = J(\boldsymbol{u}) + \big(J'(\boldsymbol{u}), \boldsymbol{u}_\varepsilon - \boldsymbol{u} \big)_M + o(\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}\|)$$

$\big(J'(\boldsymbol{u}), \boldsymbol{w} \big)_M = \text{Re}\big(\overline{\boldsymbol{u} - \boldsymbol{u}_{\text{obs}}}, \boldsymbol{w} \big)_M$ for the least-squares case

Goal: find leading form of $\big(J'(\boldsymbol{u}), \boldsymbol{u}_\varepsilon - \boldsymbol{u} \big)_M$ as $\varepsilon \to 0$.



(a) Background (incident) problem for $\boldsymbol{u}$ (in Ω_ε):

$$\big(\boldsymbol{\sigma}[\boldsymbol{u}], \boldsymbol{\nabla}\hat{\boldsymbol{u}}_\varepsilon \big)_{\Omega_\varepsilon} - \omega^2 \big(\rho\boldsymbol{u}, \hat{\boldsymbol{u}}_\varepsilon \big)_{\Omega_\varepsilon} = F(\hat{\boldsymbol{u}}_\varepsilon) + \big(\boldsymbol{t}[\boldsymbol{u}], \hat{\boldsymbol{u}}_\varepsilon \big)_{s_{\varepsilon,z}}$$

(b) Scattering problem for $\boldsymbol{u}_\varepsilon$:

$$\big(\boldsymbol{\sigma}[\boldsymbol{u}_\varepsilon], \boldsymbol{\nabla}\hat{\boldsymbol{u}}_\varepsilon \big)_{\Omega_\varepsilon} - \omega^2 \big(\rho\boldsymbol{u}_\varepsilon, \hat{\boldsymbol{u}}_\varepsilon \big)_{\Omega_\varepsilon} = F(\hat{\boldsymbol{u}}_\varepsilon)$$

(c) Adjoint problem:

$$\big(\boldsymbol{\sigma}[\hat{\boldsymbol{u}}_\varepsilon], \boldsymbol{\nabla}(\boldsymbol{u}_\varepsilon - \boldsymbol{u}) \big)_{\Omega_\varepsilon} - \omega^2 \big(\rho\hat{\boldsymbol{u}}_\varepsilon, \boldsymbol{u}_\varepsilon - \boldsymbol{u} \big)_{\Omega_\varepsilon} = -\big(J'(\boldsymbol{u}), \boldsymbol{u}_\varepsilon - \boldsymbol{u} \big)_M$$

$$\begin{aligned}\big(J'(\boldsymbol{u}), \boldsymbol{u}_\varepsilon - \boldsymbol{u} \big)_M &= \text{Re}\Big\{ \big(\boldsymbol{t}[\boldsymbol{u}], \hat{\boldsymbol{u}}_\varepsilon \big)_{s_{\varepsilon,z}} \Big\} \\ &= \varepsilon^3 |\mathcal{V}| \text{Re}\Big\{ \boldsymbol{\sigma}[\boldsymbol{u}] : \boldsymbol{\mathcal{A}} : \boldsymbol{\sigma}[\hat{\boldsymbol{u}}] - \omega^2 \rho \boldsymbol{u} \cdot \hat{\boldsymbol{u}} \Big\}(\boldsymbol{z}) + o(\varepsilon^3) \quad \text{(using expansion of } \hat{\boldsymbol{u}}_\varepsilon - \hat{\boldsymbol{u}} \text{ on } s_{\varepsilon,z})\end{aligned}$$

Topological derivative of $J(\boldsymbol{u}_\mathsf{V}) = \mathcal{J}(V)$ at $\boldsymbol{u}_\mathsf{V} = \boldsymbol{u}$:

$$J(\boldsymbol{u}_\varepsilon) = J(\boldsymbol{u}) + \varepsilon^3 \mathcal{T}(\boldsymbol{z}) + o(\varepsilon^3), \qquad \mathcal{T}(\boldsymbol{z}) = |B_\varepsilon| \, \text{Re}\Big(\boldsymbol{\sigma}[\boldsymbol{u}] : \boldsymbol{\mathcal{A}} : \boldsymbol{\sigma}[\hat{\boldsymbol{u}}] - \omega^2 \rho \boldsymbol{u} \cdot \hat{\boldsymbol{u}} \Big)(\boldsymbol{z})$$

Conclusion and outlook

- Adaptation of methods previously used to obtain asymptotic expansions for **embedded** objects;
- Generic methodology, applies to other contexts (potential, acoustics...)
- Asymptotic model allows to formulate topological derivatives on S

Work in progress:

- Justification of solution expansion: ongoing
- Practical comparisons with (numerical or experimental) reference solutions: to do

Extensions

- (partially) filled indentations (e.g. material resulting from corrosion)
- Emerging cracks
- Higher-order (in ε) asymptotic models
- Transient case

Thank you for your kind attention!

Any questions?