A VORTICITY CREATION ALGORITHM FOR THE NAVIER-STOKES EQUATIONS IN ARBITRARY DOMAIN

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ABSTRACT

We propose an algorithm of vorticity creation type for the Navier-Stokes equation in arbitrary two dimensional domains. This algorithm can be seen as a rephrasing of Chorin's popular algorithm and one can prove its convergence in the linear case under the sole assumption that the initial enstrophy is finite. The convergence proof follows from delicate energy estimates.

1. INTRODUCTION

The problem of vorticity boundary conditions for the incompressible two dimensional Navier-Stokes equations is a challenging mathematical and numerical question.

On the one hand the value of the vorticity at the boundary of obstacles is a physically relevant quantity. The vorticity, the generation of which is very often restricted at boundaries, is crucial in the understanding of most of the features of the flow (e.g. boundary layers, wakes, small scales or large eddies). This is one of the motivation for the use of numerical algorithms based on the vorticity formulation of the equations, in particular vortex methods (Chorin, 1973). For these algorithms the determination of correct -exact or at least approximate- vorticity boundary condtions is clearly essential.

On the other hand, the value of the vorticity at boundaries is not directly accessible form the mathematical model where, most often, boundary conditions are given in terms of the velocity, for instance the so-called no-slip condition on which this paper will focus.

From the mathematical point of view, the need to prescribe vorticity values at the boundary stems form the fact that, in absence of such boundary condition the convection diffusion equation on the vorticity is underdetermined, while, in the system allowing to compute the divergence free velocity in terms of the vorticity, enforcing values of both normal and tangential components of the velocity would lead to an overdertermined problem. The issue is therefore to determine the right boundary condition, either in the Dirichlet or Neumann form, to prescribe on the vorticity, such that, the vorticity solution to the convection-diffusion equation is at all time admissible in the sense that the related velocity with zero normal component has also zero tangential component. The Neumann form of this boundary condition is sometimes prefered because it is directly interpretable in terms of local vorticity production at the boundary.

There are several ways to construct exact vorticity boundary conditions. One, proposed by Anderson (1989), is to compute the time derivative of the tangential derivative of the velocity in terms of the time derivative fo the vorticity in the domain. Using the advection diffusion equation satisfied by the vorticity and integrating by parts lead to an integral-differential equation for the normal derivative of the vorticity at the boundary. Another way, proposed in Cottet (1988), is to differentiate the system giving the velocity in terms of the vorticity by writing $-\Delta u = \mathrm{curl}\,\omega$. This system is now naturally completed by the values of both components of the velocity at the boundary. The role of the vorticity boundary condition is then to guarantee that the velocity is divergence free, or equivalently that $\omega = \mathrm{curl}\,u$, by enforcing this last identity at the boundary.

All these formulations involve a delicate coupling of the velocity and vorticity through boundary conditions and equations inside the computational domain. It is worth noticing that, although they are equivalent to the Navier-Stokes equations in the original velocity-pressure formulation and that numerical methods based on these formulations have been succesfully implemented, it is not clearly understood under which conditions natural discretizations of the equations will lead to stable numerical algorithms. The reason is that, to our knowledge, no energy estimates are available for these models. Energy estimates, which will play a central role in this paper, are crucial both to check the well-posedness of the continous models, and to ensure that numerical or round-off errors in the computation of the velocity will not dramatically affect the computation of the vorticity at the boundary and ultimately, through the convection diffusion equations in the domain, the whole vorticity field.

As a matter of fact, we only know one complete numerical algorithm based on the vorticity formulation of the Navier-Stokes equation, for which convergence has been proved (Hou and Wetton, 1992). This is a finite-difference algorithm using a second order approximate Dirichlet vorticity boundary condition. The stability of the algorithm in the energy norm for the velocity comes from a fortunate sign in the definition of this algorithm.

Our primary goal here was to understand the convergence properties of Chorin's vorticity creation. Rather than trying to find explicitely a correct vorticity boundary condition, the idea in this algorithm is based on the physical intuition of the mechanisms which are underlying vorticity creation. It involves a splitting of the system into 3 substeps. In the 2 first, the Navier-Stokes equations are solved without any vorticity production at the boundary, which means homogeneous Neumann boundary conditions. In the third step, a vortex sheet is introduced at the boundary in order to cancel the slip resulting from the 2 previous steps. This vortex sheet can be interpreted as the single layer potential necessary for the computation of the stream function associated to the vorticity. The difficulty in the convergence analysis of this scheme is that it does not clearly result from the implementation of a given vorticity boundary condition -either on Dirichlet or Neumann form- and, as a result, it is difficult to define it in an intrinsic way, independent of the geometry of the domain. As a matter of fact, the only convergence proof so far for this algorithm (Benfatto and Pulvirenti, 1986) is restricted to the half plane case and is heavily based on extensions of vorticity over the entire plane and the use of integral representations in the plane. Note that, to our knowledge, even for the Stokes problem there is no convergence proof valid for more general geometries. For further discussions on splitting and vorticity creation formulas we refer to Beale and Chorin, Hugues, Mac-Cracken and Marsden (1978).

In Cottet (1988), we suggested an alternative formulation of Chorin's algorithm based on some duality relations between normal derivatives of the vorticity and time derivatives of the tangential component of the velocity. In this paper, we explore further this direction and propose a splitting based vorticity creation algorithm which can be formulated in any geometry. In the half-plane case, this algorithm leads to Chorin's algorithm through one additional natural time-discretization.

We first define, in section 2, our algorithm for the Stokes case. We prove that this algorithm is enstrophy decreasing, and use ensergy estimates to prove its convergence. We then turn in section 3 to the Navier-Stokes equation. In this case we were not able to prove energy estimates similar to the ones valid for the Stokes case. One can actually not expect the algorithm to be enstrophy decreasing, since this property is not true for the continuous Navier-Stokes equations. Energy estimates on the vorticity are rather subtle even in the continuous case, and it seems difficult to reproduce them in the discrete case. However we show that, if we assume the same stability properties for the algorithm as in the linear case, convergence towards the exact solution follows essentially from the same lines.

2. THE STOKES PROBLEM

As already stressed above our analysis will be possible due to the reformulation of this algorithm in an intrinsic way, that is a way which does not make use of any

mapping of the domain or extension of the vorticity outside the domain.
In their vorticity formulation the Stokes equations read

$$\frac{\partial\omega}{\partial t} - \Delta\omega = 0 \text{ in } \Omega \tag{1}$$

$$\omega(\cdot,0) = \omega_0 \text{ for } x \in \Omega \text{ and } t \in [0,T] \tag{2}$$

$$\frac{\partial\omega}{\partial\nu} = g \text{ on } \partial\Omega \tag{3}$$

where the (unknown) boundary condition g is such that for every time t the solution to the elliptic system

$$-\Delta\psi = \omega \text{ in } \Omega \tag{4}$$

$$\psi = 0 \text{ on } \partial\Omega \tag{5}$$

also satisfies

$$u \cdot \tau = -\frac{\partial\psi}{\partial\nu} = 0 \text{ on } \partial\Omega \tag{6}$$

In the above equations ν denotes the normal to $\partial\Omega$ oriented outward, and τ is the tangent to $\partial\Omega$ (such that (ν,τ) defines the positive orientation of the plane). The velocity field u will be said to be associated to ω, in the sense that $u = \operatorname{curl}\psi$, where ψ statisfies (4)-(5). For simplicity we will assume that the domain Ω is bounded and with smooth boundary.

Let h be a positive number and define $t_n = nh$ for $0 \le n \le N$, with $Nh = T$. In our formulation, the algorithm consists in recursively defining two vorticity fields ω_1^h, ω_2^h and a sequence ω_n in the following way: first we initialize the sequence ω_n by the exact initial vorticity filed, then, to obtain ω_{n+1} from ω_n we first solve, for $t \in [t_n, t_{n+1}]$

$$\frac{\partial\omega_1^h}{\partial t} - \Delta\omega_1^h = 0 \text{ in } \Omega \tag{7}$$

$$\omega_1^h(\cdot,t_n) = \omega_n \text{ in } \Omega \tag{8}$$

$$\frac{\partial\omega_1^h}{\partial\nu} = 0 \text{ on } \partial\Omega \tag{9}$$

We then define u_1^h as the velocity field associated to ω_1^h and we solve, still in the time interval $[t_n, t_{n+1}]$

$$\frac{\partial\omega_2^h}{\partial t} - \Delta\omega_2^h = 0 \text{ in } \Omega \tag{10}$$

$$\omega_2^h(\cdot,0) = \omega_n \text{ in } \Omega \tag{11}$$

$$\frac{\partial\omega_2^h}{\partial\nu} = g^h \text{ on } \partial\Omega \tag{12}$$

with

$$g^h = -\frac{\partial}{\partial t}(u_1^h \cdot \tau) \tag{13}$$

Finally we set

$$\omega_{n+1} = \omega_2^h(\cdot, t_{n+1}) \tag{14}$$

Observe that this defines a continous (with respect to time) vorticity field ω_2^h, and that the (discontinuous) vorticity ω_1^h is only used to "feed" the problem to be solved for ω_2^h with the correct boundary condition. Also notice that the value of g^h at time t_n, that is at times when u_1^h is not continuous, is not necessary to well define the algorithm (the normal derivative of the vorticity needs only be defined for almost all times).
A first important remark about this algorithm is that is produces the right net amount of vorticity: for the exact solution the mean vorticity, which is equal to the circulation of the velocity along the boundary, is zero in the no-slip case. In the algorithm, the net amount of vorticity produced by (10)-(12) is equal to the time integral of the integral along the boundary of the normal derivative of the vorticity:

$$\int_\Omega (\omega_{n+1} - \omega_n)\, dx = \int_{t_n}^{t_{n+1}} \int_{\partial\Omega} g^h \, dx \, dt.$$

But, due to (9), we have, for $t \in [t_n, t_{n+1}]$,

$$\int_\Omega \omega_1^h(x,t)\, dx = \int_\Omega \omega_n(x)\, dx$$

Thus, if the vorticity has mean value 0 at time t_n, the circulation of u_1^h remains zero for all time, and there is no net production of vorticity at the next time step.

2.1. Consistency of the algorithm with Chorin's vorticity creation algorithm in the half-space case

Let us assume in this section that

$$\Omega = \{x = (x_1, x_2) \in R^2 \ ; \ x_2 > 0\}$$

Let us focus on one step $t_n \to t_{n+1}$ of the above algorithm. We first make the following classical observation: by extending ω_2^h throughout $\partial\Omega$ in an even fashion, problem (10)-(12) is equivalent to the heat equation in the whole space with a forcing term localized on the axis $x_2 = 0$. More precisely, if one defines, for $x = (x_1, x_2) \in R^2$, $\bar\omega_2^h(x_1, x_2)$ to be $\omega_2^h(x_1, x_2)$ if $x_2 > 0$ and $\omega_2^h(x_1, -x_2)$ if not, then (10)-(12) is equivalent to

$$\frac{\partial \bar\omega_2^h}{\partial t} - \Delta \bar\omega_2^h = 2g^h(x_1) \otimes \delta(x_2) \text{ in } R^2 \tag{15}$$

A consistant time-splitting of the above equation leads to considering two substeps: one consisting of the heat equation with a zero forcing term, and the next one as a simple incrementation of the vorticity with the forcing term. The first substep is now equivalent to the Stokes equation in Ω with homogeneous Neumann boundary condition, that is (7)-(9), the result of which we denote by $\tilde{\omega}_{n+1}$. The final result of this algorithm is now

$$\omega'_{n+1} = \tilde{\omega}_{n+1} - 2\int_{t_n}^{t_{n+1}} \frac{\partial}{\partial t}(u_1^h \cdot \tau)(\cdot, t) \otimes \delta \, dt \tag{16}$$

Recall that here u_1^h is the velocity field corresponding to ω_1^h, the vorticity field solution of the heat equation with homogeneous Neumann boundary condition and inital condition ω'_n. To prove that this algorithm is actually equivalent to Chorin's algorithm, we need to show the following result.

Proposition 2.1. *Let ω_0 a smooth vorticity with zero mean value in the half plane, u_0 its velocity, with zero normal component at the boundary, and $\bar{\omega}_0$ its even extension to the whole plane. Let $\omega'_0 = \bar{\omega}_0 - 2u_0 \otimes \delta_{\partial\Omega}$, $\bar{\omega}(\cdot, t)$ the solution of the heat equation in the plane with initial condition ω'_0 and $u(\cdot, t)$ the associated velocity. Then*

$$u(\cdot, t) \cdot \tau \to 0 \ \textit{in} \ H^{-1/2}(\partial\Omega), \ \textit{as} \ t \to 0^+ \tag{17}$$

We postpone the proof of this result to the end of the paper. As a consequence, a straightforward induction argument shows that (16) can be rewritten as

$$\omega'_{n+1} = \tilde{\omega}_{n+1} - 2(u_1^h \cdot \tau)(\cdot, t_{n+1}) \tag{18}$$

This is precisely Chorin's product formula: one step of homogeneous heat equation followed by the creation of a vortex sheet at the boundary to cancel the slip at the boundary resulting from this step.

In other words we have demonstrated that, compared to our algorithm, Chorin's algorithm involves one additional, natural, time discretization.

2.2. Stability in the enstrophy norm of the algorithm

From now on we will assume that Ω is a bounded, smooth, simply connected domain. Our only assumption regarding the smoothness of the continuous solution of the Stokes equation is that the initial vorticity ω_0 is in $L^2(\Omega)$ and that the associated initial velocity vanishes at the boundary. We prove in this section that the algorithm is enstrophy decreasing. We recall that this poperty is indeed true for the continuous equations. To check it, one has to multiply the equations in the primitive variables (u, p) by the projection of Δu onto the space of divergence free vector fields with normal components vanishing at the boundary. The property follows then from routine integration by parts.

We will denote by $|\cdot|$ the L^2 norm in Ω. For simplicity in the notations we will drop everywhere the superscripts h, so that the vorticity fields produced by the algorithm are denoted by ω_1 and ω_2.

We focus here on a given time step $[t_n, t_{n+1}]$. To obtain energy estimates in this time interval, we first multiply (10) by ω_2 and integrate over Ω. We get

$$\frac{1}{2}\frac{d}{dt}|\omega_2|^2 + |\nabla\omega_2|^2 = \int_{\partial\Omega} \omega_2 \frac{\partial\omega_2}{\partial\nu}\, dx \tag{19}$$

Notice that a straightforward use of (12), combined with elliptic regularity and trace results to control the boundary term, would introduce multiplicative constants in the energy which would prevent any uniform control of the enstrophy. We thus need to estimate carefully the right hand side of (19). To this end we first observe that we can rewrite (13) as follows

$$\frac{\partial\omega_2}{\partial\nu} = \frac{\partial}{\partial t}\left(\frac{\partial\psi_1}{\partial\nu}\right) = \frac{\partial}{\partial\nu}\left(\frac{\partial\psi_1}{\partial t}\right) \text{ on } \partial\Omega$$

Moreover, by the definition of ψ_1 from ω_1 and (7):

$$\begin{aligned} -\Delta\left(\frac{\partial\psi_1}{\partial t}\right) = \frac{\partial\omega_1}{\partial t} &= \Delta\omega_1 \text{ in } \Omega &(20)\\ \frac{\partial\psi_1}{\partial t} &= 0 \text{ on } \partial\Omega &(21)\end{aligned}$$

If we set

$$\phi = \frac{\partial\psi_1}{\partial t}$$

we thus have, upon integrating by parts

$$\begin{aligned}\int_{\partial\Omega} \omega_2 \frac{\partial\omega_2}{\partial\nu}\, dx &= \int_{\partial\Omega} \omega_2 \frac{\partial\phi}{\partial\nu}\, dx = \int_\Omega \Delta\phi\, \omega_2\, dx + \int_\Omega \nabla\phi\cdot\nabla\omega_2\, dx\\ &= -\int_\Omega \Delta\omega_1\, \omega_2\, dx + \int_\Omega \nabla\phi\cdot\nabla\omega_2\, dx\\ &= \int_\Omega \nabla\omega_1\cdot\nabla\omega_2\, dx + \int_\Omega \nabla\phi\cdot\nabla\omega_2\, dx \qquad (22)\end{aligned}$$

Note that in the integration by parts leading to the last equality we have used the fact that $\partial\omega_1/\partial\nu = 0$ on $\partial\Omega$. Combining (19) and (22) yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\omega_2|^2 &= \int_\Omega \nabla\omega_2\cdot(\nabla\omega_1 - \nabla\omega_2)\, dx + \int_\Omega \nabla\phi\cdot\nabla\omega_2\, dx\\ &= \int_\Omega \nabla\omega_2\cdot\nabla e\, dx + \int_\Omega \nabla\phi\cdot\nabla\omega_1\, dx - \int_\Omega \nabla\phi\cdot\nabla e\, dx \qquad (23)\end{aligned}$$

where we have set $e = \omega_1 - \omega_2$. Next we write

$$\int_\Omega \nabla\phi\cdot\nabla\omega_1\, dx = -\int_\Omega \phi\Delta\omega_1\, dx = \int_\Omega \phi\Delta\phi\, dx = -|\nabla\phi|^2 \tag{24}$$

To establish this we used first that $\partial\omega_1/\partial\nu = 0$ on $\partial\Omega$, then that $\Delta\phi = -\Delta\omega_1$ and finally that $\phi = 0$ on $\partial\Omega$. From (23) and (24) we thus have

$$\frac{1}{2}\frac{d}{dt}|\omega_2|^2 = \int_\Omega \nabla\omega_2 \cdot \nabla e \, dx - |\nabla\phi|^2 - \int_\Omega \nabla\phi \cdot \nabla e \, dx \tag{25}$$

To recover informations on e, let us now substract (10) from (7), multiply it by e and integrate over Ω. We obtain:

$$\frac{1}{2}\frac{d}{dt}|e|^2 + |\nabla e|^2 = -\int_{\partial\Omega} e\frac{\partial\phi}{\partial\nu}\, dx$$

and arguing as for (22):

$$\frac{1}{2}\frac{d}{dt}|e|^2 + |\nabla e|^2 = -\int_\Omega \nabla\omega_1 \cdot \nabla e \, dx - \int_\Omega \nabla\phi \cdot \nabla e \, dx \tag{26}$$

Adding (25) to (26) leads to

$$\frac{1}{2}\frac{d}{dt}|\omega_2|^2 + \frac{1}{2}\frac{d}{dt}|e|^2 + |\nabla e|^2 = -|\nabla e|^2 - |\nabla\phi|^2 - 2\int_\Omega \nabla e \nabla\phi \, dx = -|\nabla e + \nabla\phi|^2 \leq 0. \tag{27}$$

Integrating this inequality over the time step $[t_n, t_{n+1}]$ and using the crucial fact that $e(\cdot, t_n + 0) = 0$, since ω_1 and ω_2 start from the same initial condition, we obtain:

$$|\omega_{n+1}|^2 - |\omega_n|^2 + |\tilde{\omega}_{n+1} - \omega_{n+1}|^2 + \int_{t_n}^{t_{n+1}} |\nabla e(\cdot, s)|^2 \, ds \leq 0 \tag{28}$$

We thus have proved, in particular, the following stability result

Theorem 2.1. *The algorithm defined by (7)-(13) is enstrophy decreasing:*

$$|\omega_n|^2 \leq |\omega_0|^2, \forall n > 0$$

2.3. Convergence towards the exact solution of the Stokes equation

Let us first sketch the main lines of the convergence arguments which will follow. First it is clear that theorem 2.2 shows that, for a subsequence, the approximate vorticity will converge weakly, in a space to be specified, to some vorticity ω. That ω is a distribution solution of the Stokes equation clearly results from the linearity of this equation. The key point is thus to prove that the velocity u associated to this vorticity ω vanishes at the boundary.

To establish this kind of property it seems at first glance that this would require the control of ω in a space where one can talk of traces, for instance $H^1(\Omega)$. Unfortunately the energy estimate (28), does not give any information of that kind. However we do have informations on the gradient of $e = \omega_1 - \omega_2$. On the other hand, (28) tells us that the series $\sum_n |e(t_n - 0)|^2$ is convergent, something which, as we will prove next, implies that the weak limit of e must be zero. We will then prove that this in turn tells us that the boundary conditions on ω_1 and ω_2 cannot differ significantly, and thus that the slip of u_1 tends to zero when $h \to 0$.

The following result summarizes the estimates available for e^h (to avoid confusion we now restore the superscript h).

Proposition 2.2. *The distributions e^h are uniformly bounded (with respect to h) in the space $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$. Furthermore we can write*

$$\frac{\partial e^h}{\partial t} = v^h + \sum_{0<nh<T} \delta(t-t_n)\alpha_n^h \tag{29}$$

where the functions v^h and the sequences (α_n^h) are respectively uniformly bounded in $L^\infty(0,T;H^{-2})$ and in $l_1 \cap l_\infty(N;H^{-2})$, and there exists a constant C such that, for all positive n

$$\|\alpha_n^h\|_{H^{-2}(\Omega)} \leq Ch \tag{30}$$

Proof: First notice that the Dirac masses in the right hand side of (29) translate the jumps that ω_1^h (and thus e^h) undergo at every time-step.

Our first claim concerning the uniform control of e^h in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ readily follows from (28). As for (29), it results from the fact that, if $t \in]t_n, t_{n+1}[$

$$\frac{\partial e^h}{\partial t} = \Delta e^h$$

Thus (29) is satisfied with $v^h = \Delta e^h$ and $\alpha_n^h = [e^h]_{|t=t_n}$.
e^h is bounded in $L^\infty(0,T;L^2(\Omega))$ and thus v^h is bounded in $L^\infty(0,T;H^{-2}(\Omega))$. Since $e^h(t) \to 0$ when $t \to t_n, t > t_n$ we can write

$$\alpha_n^h = -\int_{t_n}^{t_{n+1}} \frac{\partial e^h}{\partial t}(\cdot,s)\,ds = -\int_{t_n}^{t_{n+1}} \Delta e^h(\cdot,s)\,ds \leq h\|e^h\|_{L^\infty(0,T;H^{-2}(\Omega))}$$

from which (30) follows. $\square$

We are now in a situation where one can think of applying classical results to get a uniform control with respect to time of the family e^h. However in these results the time derivatives are in general supposed to be controled in L^p spaces rather than in measures, so we need to adapt their proof to our specific situation

Proposition 2.3. *There exists a distribution λ in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ such that, for a subsequence,*

$$e^h \to \lambda \text{ strongly in } L^\infty(0,T;L^2(\Omega)) \tag{31}$$

$$e^h \rightharpoonup \lambda \text{ weakly in } L^2(0,T;H^1(\Omega)) \tag{32}$$

Proof: We already know that there exists λ such that, upon extracting a subsequence, e^h tends to λ weakly in $L^2(0,T;H^1(\Omega))$. To simplify we will assume that $\lambda = 0$.

Let us focus on the case $t = 0$ (the other times would be dealt with through obvious modifications in the proof below).

We set $f^h(\cdot,t) = e^h(\cdot,\beta t)$, where β is a positive parameter to be specified later. If ϕ is a smooth function of time satisfying $\phi(0) = -1, \phi(T) = 0$, we can write

$$e^h(\cdot,0) = f^h(\cdot,0) = \int_0^T (\phi f^h)'(\cdot,s)\,ds$$

where of course the time derivative in the right hand side above is to be taken in a distribution sense. Using the definition of f^h and (29) we find:

$$\begin{aligned}\int_0^T (\phi f^h)'(\cdot,s)\,ds &= \int_0^T (\phi'(\cdot,s)e^h(\cdot,\beta s) + \beta\phi(\cdot,s)v^h(\cdot,\beta s)\,ds + \sum_{nh\le\beta T} \alpha_n^h \phi(\cdot,t_n/\beta)\\ &= I_1^h + I_2^h + I_3^h\end{aligned}$$

Next we write

$$\|I_2^h\|_{H^{-2}} \le C\beta^{1/2}\|v^h\|_{L^2(0,T;H^{-2})}$$

and, due to (30)

$$\|I_3^h\|_{H^{-2}} \le \|\phi\|_{L^\infty}\beta T$$

So, given $\varepsilon > 0$, we can find β small enough so that I_2^h and I_3^h are smaller than ε in the H^{-2} norm for all h . We fix this value of β. Then we observe that I_1^h is uniformly bounded in H^1 and tends to 0 weakly in H^{-2}; by the compactness of H^1 in H^{-2} this implies that, for a subsequence, I_1^h tends to 0 strongly in H^{-2}. We have thus proved that, for h small enough in a subsequence, we have

$$\|e^h(\cdot,0)\|_{H^{-2}} \le \varepsilon$$

which ends our proof. $\square$

So far we have used the informations available for e^h and its derivative. We are now going to use the remaining informations contained in the energy estimate (29) to prove that the limit λ of e^h is necessarily zero:

Proposition 2.4.

$$e^h \to 0 \text{ \textit{weakly in} } L^2(0,T;H^1(\Omega))$$

Proof: By (31) we have

$$h\sum_n \|e^h(\cdot,t_n)\|_{H^{-2}} - h\sum_n \|\lambda(\cdot,t_n)\|_{H^{-2}} \to 0 \tag{33}$$

On the other hand we know by (28) that the sum

$$\sum_{t_n \in [0,T]} \|e^h(\cdot, t_n)\|_{L^2}$$

is bounded independantly of h. Combined with (33) this implies that

$$h \sum_{t_n \in [0,T]} \|\lambda(\cdot, t_n)\|^2_{H^{-2}} \to 0 \text{ as } h \to 0.$$

Finally we notice that, by (29), $\frac{\partial \lambda}{\partial t}$ is necessarily in the space of time measures with values in $H^{-2}(\Omega)$ and thus is continuous with respect to time with values in H^{-2}. Therefore

$$h \sum_{t_n \in [0,T]} \|\lambda(\cdot, t_n)\|^2_{H^{-2}} \to \|\lambda\|^2_{L^2(0,T;H^{-2})}$$

and $\lambda = 0$. $\square$

We can now prove our convergence result

Theorem 2.2. *The sequence ω_2^h converges weakly in $L^2(0,T;L^2)$ to the exact solution of the Stokes equation as h tends to 0.*

Proof: Because of the energy estimate (28), we know that there exists a vorticity field ω such that, for a subsequence, w_2^h converges weakly to ω in $L^2(0,T;L^2)$. Since ω_2^h satisfies (10) for almost every time, it is clear that (1) is satisfied in the sense of distribution. It only remains to prove that the velocity field u related to ω (still in the sense that $\operatorname{curl} u = \omega$ in Ω and $u \cdot \nu = 0$ on $\partial\Omega$) vanishes at the boundary. By (7) and (10) and the fact that we assumed Ω simply connected, we can find 2 pressure fields p_1^h and p_2^h such that

$$\frac{\partial u_1^h}{\partial t} - \Delta u_1^h = \nabla p_1^h \;;\; \frac{\partial u_2^h}{\partial t} - \Delta u_2^h = \nabla p_2^h$$

Using obvious vector identities together with (9),(12) we get

$$\frac{\partial}{\partial \tau}(p_1^h - p_2^h) = -\frac{\partial}{\partial t}(u_2^h \cdot \tau) \;;\; \frac{\partial}{\partial \nu}(p_1^h - p_2^h) = -\frac{\partial}{\partial \tau}(\omega_1^h - \omega_2^h) \tag{34}$$

In passing, let us emphasize that these identities, which somehow summarize all the informations contained in the algorithm, are essential in understanding how normal derivatives of the vorticity are naturally linked to time derivatives of tangential velocities.

Let now ϕ denote a smooth test function on $\partial\Omega$ and consider the (smooth) function ψ defined on Ω as the solution to the following problem

$$\Delta \psi = 0 \text{ in } \Omega \;;\; \frac{\partial \psi}{\partial \nu} = \frac{\partial \phi}{\partial \tau} \text{ on } \partial\Omega$$

Green's formula along with (34) enables us to write

$$\begin{aligned}\int_{\partial\Omega} \frac{\partial}{\partial t}(u_2^h \cdot \tau)\phi\, dx &= \int_{\partial\Omega}(p_1^h - p_2^h)\frac{\partial \phi}{\partial \tau}\, dx = -\int_{\partial\Omega} \frac{\partial}{\partial \nu}(p_1^h - p_2^h)\psi\, dx \\ &= -\int_{\partial\Omega}(\omega_1^h - \omega_2^h)\frac{\partial \psi}{\partial \tau}\, dx \end{aligned} \tag{35}$$

But we have proved in propositions 2.3 and 2.4 that $\omega_1^h - \omega_2^h$ tends to 0 weakly in $L^2(0,T;H^1(\Omega))$. Therefore its trace on $\partial\Omega$ tends to 0 weakly in $L^2(0,T;H^{1/2}(\partial\Omega))$. Due to (35) we deduce that $\frac{\partial}{\partial t}u_2^h \cdot \tau$ tends to 0 in $L^2(0,T;\mathcal{D}'(\partial\Omega))$. Since $u_2^h \cdot \tau(\cdot,0) = 0$ (u_2^h is at time 0 the velocity associated to the exact initial vorticity field), this implies that

$$u_2^h \cdot \tau \to 0 \text{ in } L^2(0,T;\mathcal{D}'(\partial\Omega))$$

The limit u of u_2^h has thus to vanish at the boundary. Finally the uniqueness of all possible limits of subsequences proves that the whole sequence ω_2^h tends to the exact solution of the Stokes equation. □

3. THE NAVIER-STOKES SYSTEM

To define our vorticity creation algorithm we set, as for the linear case, $t_n = nh$. We define two vorticity fields ω_1^h and ω_2^h and a sequence ω_n by solving in $]t_n, t_{n+1}[$, first

$$\frac{\partial \omega_1^h}{\partial t} + \mathrm{div}(u_1^h \omega_1^h) - \Delta\omega_1^h = 0 \text{ in } \Omega \tag{36}$$

$$\omega_1^h(\cdot, t_n) = \omega_n \text{ in } \Omega \tag{37}$$

$$u_1^h = \mathrm{curl}\,(-\Delta_d^{-1}(\omega_1^h)) \text{ in } \Omega \tag{38}$$

$$\frac{\partial \omega_1^h}{\partial \nu} = 0 \text{ on } \partial\Omega \tag{39}$$

In (38), Δ_d^{-1} denotes the solution operator for the Laplace equation in Ω with homogeneous Dirichlet boundary condition on $\partial\Omega$. We next define ω_2^h as the solution of

$$\frac{\partial \omega_2^h}{\partial t} + \mathrm{div}(u_1^h \omega_2^h) - \Delta\omega_2^h = 0 \text{ in } \Omega \tag{40}$$

$$\omega_2^h(\cdot, t_n) = \omega_n \text{ in } \Omega \tag{41}$$

$$\frac{\partial \omega_2^h}{\partial \nu} = g^h \text{ on } \partial\Omega \tag{42}$$

with

$$g^h = [\frac{\partial u_1^h}{\partial t} + (u_1^h \cdot \nabla)u_1^h] \cdot \tau \tag{43}$$

Observe that we have chosen to use the same velocity field u_1^h in the convection-diffusion systems giving ω_1^h and ω_2^h, but this is probably not crucial.

The verification that the algorithm creates the right net amount of vorticity follows from the same argument as for the Stokes case (because of the conservative form of the vorticity equations, the non linear terms in (36),(38) do not contribute to this).

3.1. Recovering Chorin's algorithm

Assume that Ω is the half plane. If we argue as in section 2.1 and split into a free convection-diffusion equation (which itself can be split into a convection step with no through flow condition, then a diffusion step with homogeneous Neumann boundary condition) followed by a forcing step at the boundary, we are led to the following natural scheme

$$\omega'_{n+1} = \tilde{\omega}_{n+1} - 2\left[\int_{t_n}^{t_{n+1}} (\frac{\partial u_1^h}{\partial t} + (u_1^h \cdot \nabla)u_1^h)\, ds\right] \cdot \tau \otimes \delta \tag{44}$$

where we have denoted by $\tilde{\omega}_{n+1}$ the value at time t_{n+1} of ω_1^h. Let us now set $U_1^h = \frac{\partial u_1^h}{\partial t} + (u_1^h \cdot \nabla)u_1^h$ and denote by $X_1^h(t; x, s)$ the value at time t of the characteristics associated to the velocity u_1^h and passing at time s through x. We make the following observation: if $t \in [t_n, t_{n+1}]$ then $X_1^h(t; x, t_{n+1}) - x \to 0$ as $h \to 0$ as soon as u_1^h is smooth enough. Thus, under suitable smoothness assumptions:

$$U_1^h(X_1^h(t; x, t_{n+1}), t) - U_1^h(x, t) = O(\Delta t)$$

if $t \in [t_n, t_{n+1}]$. On the other hand we can write

$$U_1^h(X_1^h(t; x, t_{n+1}), t) = \frac{\partial}{\partial t} u_1^h(X_1^h(t; x, t_{n+1}), t)$$

If $x \in \partial\Omega$, $X_1^h(t; x, t_{n+1})$ remains on $\partial\Omega$ for all time since $u_1^h \cdot \nu = 0$ on $\partial\Omega$. It thus follows from what precedes that (43) is consistent with the following scheme:

$$\omega'_{n+1} = \tilde{\omega}_{n+1} - 2\left[\int_{t_n}^{t_{n+1}} (\frac{\partial}{\partial t}(u_1^h(X_1^h(t; x, t_{n+1}), t)\, ds\right] \cdot \tau \otimes \delta \tag{45}$$

Finally it can be proven by arguments similar to those used for the Stokes equation that the effect of a free convection diffusion step is to immediately cancel the slip at the boundary, so (45) can be rewritten as

$$\omega'_{n+1} = \tilde{\omega}_{n+1} - 2u_1^h(\cdot, t_{n+1}) \cdot \tau \otimes \delta$$

The conclusion is that, as in the Stokes case, although in a more involved way, Chorin's vorticity creation algorithm can be interpreted as an additional time discretization over our algorithm.

3.2. Consistency of the algorithm

As we already said, stability estimates seem hard to establish in the non linear case. However, in order to validate the algorithm, it is worthwhile to check that, if we assume that a priori estimates similar to (28) are valid, convergence of the algorithm follows.

We will thus assume that there exists a constant M such that for all $n > 0$

$$|\omega_n|^2 + \sum_{k=0}^{n} |\omega_k - \tilde{\omega}_k|^2 + \int_0^{t_n} |\nabla e(\cdot, s)|^2 \, ds \leq M \tag{46}$$

As for the linear case, one important consequence of the energy estimate (46) is the fact that e^h tends to 0. This follows from Proposition 2.3 which extends without modification to the non linear case. To check it we write, for $t \in]t_n, t_{n+1}[$,

$$\frac{\partial e^h}{\partial t} = \Delta e^h - \mathrm{div}\,(u_1^h e^h)$$

It is readily seen from (36)-(38) that, for $t \in [t_n, t_{n+1}]$, $\|\omega_1^h(\cdot, t)\|_{L^2} \leq \|\omega_n\|_{L^2} \leq M$. By ellpitic regularity u_1^h is therefore bounded in $L^\infty(0, T; L^p(\Omega))$ for all finite p. So $u_1^h e^h$ is bounded in, say, $L^\infty(0, T; L^{3/2}(\Omega)) \subset L^\infty(0, T; H^{-1}(\Omega))$, and $\mathrm{div}\,(u_1^h e^h)$ is bounded in $L^\infty(0, T; H^{-2}(\Omega))$. The decomposition (29) and the estimate (30) follows then from the same arguments as in the linear case. Propositions 2.4 and 2.5 in turn remain true, as they are only based on these 2 results and on the fact that the series $\sum_n \|e^h(\cdot, t_n - 0)\|_{L^2}$ are uniformly bounded .

One can then prove

Theorem 3.3. *Assume (46). The sequence ω_2^h converges weakly in $L^2(0, T; L^2(\Omega))$ towards the solution ω of the Navier-Stokes equation. Morevoer the velocities u_2^h converge strongly to the exact velocity in $L^2(0, T; L^p(\Omega))$ for all finite p.*

Proof: Let us first summarize the estimates available for (u_2^h, ω_2^h). By (46)

$$u_2^h \text{ is bounded in } L^\infty(0, T; H^1(\Omega))$$

By (29) and the fact already used that u_1^h is bounded in $L^\infty(0, T; L^p(\Omega))$ for all finite p, we also have that

$$\frac{\partial u_2^h}{\partial t} \text{ is bounded in } L^\infty(0, T; H^{-2}(\Omega))$$

For all finite p, $H^1(\Omega)$ is compactly imbedded in $L^p(\Omega)$, which is imbedded in $H^{-2}(\Omega)$. By classical compactness arguments we deduce that there exists a velocity u, such that, for a subsequence,

$$u_2^h \to u \text{ strongly in } L^\infty(0, T; L^p(\Omega))$$

for all finite p. But weak compactness also shows that, still for a subsequence, ω_2^h tends to $\omega = \operatorname{curl} u$ weakly in $L^\infty(0,T;L^2(\Omega))$ and thus

$$u_2^h \omega_2^h \rightharpoonup u\omega \text{ weakly in } L^\infty(0,T;L^{3/2}(\Omega)).$$

This first shows that (u,ω) is solution of the Navier-Stokes equation in Ω. As for the linear case, the key point is now to check that u vanishes at the boundary. We proceed as for the Stokes problem, with however 2 slight modifications. The first one is motivated by the fact that we used, for the non linear term in the definition of ω_2^h, u_1^h and not u_2^h. This prevents us from writing $(u_1^h \cdot \nabla)\omega_2^h$ on a rotational form.

Let

$$v_1^h = \operatorname{curl}\left[(u_1^h \cdot \nabla)u_2^h\right] - (u_1^h \cdot \nabla)\omega_2^h$$

and define ϕ_1^h as the solution to the elliptic problem

$$\Delta \phi_1^h = v_1^h \text{ in } \Omega \; ; \; \frac{\partial \phi_1^h}{\partial \nu} = 0 \text{ on } \partial\Omega$$

We claim that

$$\phi_1^h \to 0 \text{ in } L^2(0,T;W^{1,4/3}(\Omega)) \tag{47}$$

The reason is that, since $\operatorname{curl}\left[(u_2^h \cdot \nabla)u_2^h\right] = (u_2^h \cdot \nabla)\omega_2^h$, we can rewrite

$$v_1^h = \operatorname{curl}\left[((u_1^h - u_2^h)\cdot \nabla)u_2^h\right] - ((u_1^h - u_2^h)\cdot\nabla)\omega_2^h$$

and thus

$$\|v_1^h\|_{W^{-1,4}} \le C\|u_1^h - u_2^h\|_{L^4}\|\omega_2^h\|_{L^2}$$

where $W^{-1,4}$ denotes the topological dual of $W^{1,4/3}(\Omega)$. In what precedes, we have proved that $e^h \rightharpoonup 0$ weakly in $L^2(0,T;H^1(\Omega))$. By compactness of $H^1(\Omega)$ into $L^4(\Omega)$ this implies that

$$u_1^h - u_2^h \to 0 \text{ in } L^2(0,T;L^4(\Omega))$$

Therefore $v_1^h \to 0$ strongly in $L^2(0,T;W^{-1,4}(\Omega))$ which proves (47).

We now come back to our velocities u_1^h and u_2^h. We have

$$\begin{aligned}
\operatorname{curl}\left[\tfrac{\partial u_1^h}{\partial t} + (u_1^h\cdot\nabla)u_1^h - \Delta u_1^h\right] &= 0 \\
\operatorname{curl}\left[\tfrac{\partial u_2^h}{\partial t} + (u_1^h\cdot\nabla)u_2^h - \Delta u_2^h - \operatorname{curl}\phi_1^h\right] &= 0
\end{aligned}$$

This yields 2 pressure fields p_1^h and p_2^h such that

$$\begin{aligned}
\frac{\partial u_1^h}{\partial t} + (u_1^h\cdot\nabla)u_1^h - \Delta u_1^h &= \nabla p_1^h \\
\frac{\partial u_2^h}{\partial t} + (u_1^h\cdot\nabla)u_2^h - \Delta u_2^h - \operatorname{curl}\phi_1^h &= \nabla p_2^h
\end{aligned}$$

We have

$$\Delta(p_1^h - p_2^h) = \mathrm{div}\,[(u_1^h \cdot \nabla)(u_1^h - u_2^h)]$$

Using (43) we get

$$\begin{aligned}
\frac{\partial}{\partial \tau}(p_1^h - p_2^h) &= -\frac{\partial}{\partial t}(u_2^h \cdot \tau) + (u_1^h \cdot \nabla)u_2^h \cdot \tau \\
\frac{\partial}{\partial \nu}(p_1^h - p_2^h) &= -\frac{\partial}{\partial \tau}(\omega_1^h - \omega_2^h) + \frac{\partial \phi_1^h}{\partial \tau} + ((u_1^h \cdot \nabla)(u_1^h - u_2^h)) \cdot \nu
\end{aligned}$$

from which we deduce that, for a given smooth test function ϕ and if ψ denotes the solution of $\Delta\psi = 0$ in Ω, $\partial\psi/\partial\nu = \partial\phi/\partial\tau$ on $\partial\Omega$

$$\int_{\partial\Omega} [\frac{\partial u_2^h}{\partial t} - (u_1^h \cdot \nabla)u_2^h] \cdot \tau\phi\, dx = \int_{\partial\Omega} (p_1^h - p_2^h)\frac{\partial \phi}{\partial \tau}\, dx = \int_{\partial\Omega} \frac{\partial \psi}{\partial \nu}(p_1^h - p_2^h)\, dx$$

Integrations by parts yield

$$\begin{aligned}
&\int_{\partial\Omega} \frac{\partial \psi}{\partial \nu}(p_1^h - p_2^h)\, dx = \\
&= \int_\Omega \Delta\psi(p_1^h - p_2^h)\, dx + \int_\Omega \nabla\psi\nabla(p_1^h - p_2^h)\, dx \\
&= -\int_\Omega \psi\Delta(p_1^h - p_2^h)\, dx + \int_{\partial\Omega} \psi\frac{\partial}{\partial \nu}(p_1^h - p_2^h)\, dx \\
&= -\int_\Omega \psi \mathrm{div}\,[(u_1^h \cdot \nabla)(u_1^h - u_2^h)]\, dx + \int_{\partial\Omega} \psi\frac{\partial}{\partial \nu}(p_1^h - p_2^h)\, dx \\
&= \int_\Omega \nabla\psi \cdot [(u_1^h \cdot \nabla)(u_1^h - u_2^h)]\, dx + \int_{\partial\Omega} \psi[\frac{\partial}{\partial \nu}(p_1^h - p_2^h) - (u_1^h \cdot \nabla)(u_1^h - u_2^h) \cdot \nu]\, dx \\
&= \int_\Omega \nabla\psi \cdot [(u_1^h \cdot \nabla)(u_1^h - u_2^h)]\, dx + \int_{\partial\Omega} (-\phi_1^h + \omega_1^h - \omega_2^h)\frac{\partial \psi}{\partial \tau}\, dx \qquad (48)
\end{aligned}$$

But, on the one hand, we know that, as in the linear case, $\omega_1^h - \omega_2^h$ tends to 0 weakly in $L^2(0,T;H^{1/2})$, and, on the other hand, we deduce from (47) that ϕ_1^h tends to 0 in $L^2(0,T;W^{7/12,4/3})$ ($W^{7/12,4/3}$ is the trace space associated to $W^{1,4/3}$). Whence we obtain that the boundary integral in the right hand side above tends to 0. We also have

$$\|(u_1^h \cdot \nabla)(u_1^h - u_2^h)\|_{L^1} \leq C|u_1^h|_2|e^h|_2$$

By (31), this implies that the first integral tends to zero. Passsing to the limit in (48) thus shows that the scalar function $\gamma = u \cdot \tau$ is solution to the following equation:

$$\frac{\partial \gamma}{\partial t} - \frac{1}{2}\frac{\partial(\gamma^2)}{\partial \tau} = 0$$

Since the initial velocity, corresponding to the exact vorticity field ω_0, vanishes at the boundary, this is enough to ensure that γ remains zero for all time, and our proof is completed. $\square$

4. PROOF OF PROPOSITION 2.1

Let us consider the function $\psi'(\cdot,t)$ solution to

$$-\Delta\psi' = \bar{\omega} \text{ in } \Omega\ ;\ \frac{\partial\psi'}{\partial\nu} = 0 \text{ on } \partial\Omega \tag{49}$$

and let us prove

Lemma 4.1.

$$\psi'(\cdot,t) \to 0 \ \textit{in}\ H^{1/2}(\partial\Omega)\ \textit{as}\ t \to 0^+ \tag{50}$$

We claim that this is actually equivalent to the result we are looking for in Proposition 2.1: if we write $u = \operatorname{curl}\psi$ with

$$-\Delta\psi = \bar{\omega} \text{ in } \Omega\ ;\ \psi = 0 \text{ on } \partial\Omega$$

and, if we assume that (50) is true, then $\psi" = \psi - \psi'$ satisfies

$$-\Delta\psi" = 0 \text{ in } \Omega\ ;\ \psi"(\cdot,t) \to 0 \text{ in } H^{1/2}(\partial\Omega)$$

This immediately yields

$$\frac{\partial\psi"}{\partial\nu} \to 0 \text{ in } H^{-1/2}(\partial\Omega)$$

But we precisely have $u\cdot\tau = -\frac{\partial\psi"}{\partial\nu}$ and our result follows. □

To prove (50), let us first show that, when $t \to 0^+$

$$\bar{\omega}(\cdot,t) \to \bar{\omega}_0 - 2u_0 \otimes \delta_{\partial\Omega} \text{ in } H^{-1}(R^2). \tag{51}$$

Observe that, since ω_0 has zero mean value in R^2, u_0 is in $H^1(R^2)$ and its trace on $\partial\Omega$ is therefore in particular in $L^2(\partial\Omega)$. It is then readily seen that $\bar{\omega}_0 - 2u_0 \otimes \delta_{\partial\Omega}$ is in $H^{-1}(R^2)$ so that (52) is actually nothing but the continuity with respect to time of the solution of the heat equation in $H^{-1}(R^2)$. The proof of this result can be easily derived using the Fourier transform. If λ denotes the Fourier transform of $\bar{\omega}_0 - 2u_0 \otimes \delta_{\partial\Omega}$, then $\hat{\omega}(\cdot,t)$, Fourier transform of $\bar{\omega}(\cdot,t)$ is given by

$$\hat{\omega}(\xi,t) = e^{-|\xi|^2 t}\lambda(\xi)$$

and, for $t \to 0^+$,

$$\|(1+|\xi|^2)^{-1/2}[\hat{\omega}(\xi,t) - \lambda(\xi)]\|_{L^2_\xi} \to 0$$

by Lebesgue's theorem, which yields (51).

Let us finally prove (50). We first observe that the solution to (49) can be extended in an even way across $\partial\Omega$, and thus can be written as

$$\psi' = G * \bar{\omega}$$

where $G(x) = -(2\pi)^{-1}\log|x|$ is the elementary solution for the Laplace equation in 2-D. By (51), this implies that

$$\psi'(\cdot,t) \to G * [\bar{\omega}_0 - 2u_0 \otimes \delta_{\partial\Omega}] \text{ in } H^{1/2}(R^2)$$

But $u_0 \cdot \tau = -\partial\psi_0/\partial\nu$, where ψ_0 is the stream function associated to u_0, so that, upon developping the integrals in the above convolution, it is readily seen that it yields the integral representation of ψ_0. Since $\psi_0 = 0$ on $\partial\Omega$, we in particular get (50).

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