CONVERGENCE OF PARTICLE METHODS WITH RANDOM REZONING FOR THE 2-D EULER AND NAVIER-STOKES EQUATIONS

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Abstract. We prove the convergence of a vortex method for the 2d Euler and Navier-Stokes equations when only weak solutions are available. The approximation uses random rezoning at each time step and the convergence is obtained provided the various discretization parameters are linked by some power laws. The techniques rely on compactness properties similar to those giving the existence of weak solutions.

1. Introduction. The goal of this paper is to show the convergence of particle methods, based on random rezoning of the particles at each time step, in the regime where exact solutions are not smooth and only compactness arguments are available to prove their existence. Our main example is the two dimensional Euler equations of inviscid incompressible fluids when the initial vorticity field is an L^p $(4/3 < p < +\infty)$ function. Then the Euler equations have only weak solutions and, to the best of our knowledge, there is no proof of convergence, in such a case, for the vortex method (cf. [11, 16] for a review and [2, 5, 6, 15]), except the partial results contained in [1, 4, 8, 9]. In the present paper, a vortex type method is designed for the unit square with periodic boundary condition $X = R^2/Z^2$ in the following way : at time $t = n\Delta t$ (where $\Delta t > 0$ denotes the time step), the vorticity field $\omega(t, \cdot)$ is approximated by a finite sum of N vortex blobs located at $x_{n,i}$ with weight $\alpha_{n,i}$ for $i = 1, \cdots, N$, namely

$$\Omega_n(x) = \frac{1}{N}\sum_{i=1}^{N} \alpha_{n,i}\zeta_\varepsilon(x - x_{n,i}) - m_n = \frac{1}{N}\sum_{i=1}^{N} \alpha_{n,i}[\zeta_\varepsilon(x - x_{n,i}) - 1],$$

where $\varepsilon > 0$ is a cutoff-parameter, ζ_ε a smooth approximation to the delta function and m_n is a constant chosen so that the discrete vorticity field has zero mean.

The corresponding velocity field u is computed by solving the two dimensional periodic Poisson problem

$$\mathrm{curl} u = \Omega_n \quad , \quad \mathrm{div} u = 0,$$

or, in short,

$$u = \mathrm{curl}(-\Delta)^{-1}\Omega_n.$$

(Notice that the solvability condition for this periodic boundary value problem is precisely that Ω_n has zero mean.)

Next, a random sequence of N points independently and uniformly distributed in X, $(x_{n+1/2,i}, i = 1, \cdots, N)$, is generated and a new distribution of blobs is obtained,

$$\tilde{\Omega}_n(x) = \frac{1}{N}\sum_{i=1}^{N} \alpha_{n+1,i}[\zeta_\varepsilon(x - x_{n+1/2,i}) - 1],$$

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by setting

$$\alpha_{n+1,i} = \Omega_n(x_{n+1/2,i}) = \frac{1}{N}\sum_{j=1}^{N} \alpha_{n,j}[\zeta_\varepsilon(x_{n+1/2,i} - x_{n,j}) - 1].$$

Then, for each $i = 1, \cdots, N$, the ordinary differential equation

$$\frac{dx}{dt} = u(x),$$

is solved, for $n\Delta t \leq t \leq (n+1)\Delta t$, with initial condition $x(n\Delta t) = x_{n+1/2,i}$ and the time-independant velocity field u computed from Ω_n as above indicated. The result at time $(n+1)\Delta t$ is denoted by $x_{n+1,i}$ which leads to the updated blob distribution

$$\Omega_{n+1}(x) = \frac{1}{N}\sum_{i=1}^{N} \alpha_{n+1,i}[\zeta_\varepsilon(x - x_{n+1,i}) - 1].$$

Notice that this method is not the well known 'random vortex method' for the Navier-Stokes equations [5], [15], [10], [14]..., where vortex particles are alternately advected by the velocity field and displaced by a random walk. Here randomness is introduced by resampling the vortex particles. Thanks to this 'random rezoning', we can prove that, for an adequate choice of the discretization parameters Δt, N, ε (typically $N >> \varepsilon^{-4}$, for $\Delta t = 0(\varepsilon)$, to be compared with $N >> \varepsilon^{-2}$ in the smooth regime ([2])), the discrete vorticity stays almost surely uniformly bounded in $L^p(X)$, which enforces, by standard compactness arguments, that the discrete velocity field has a convergent subsequence toward a weak solution to the Euler equations (or Navier-Stokes equations for a different scaling of the discretization parameters) on R^2/Z^2. The L^p bound is obtained by shortcutting the traditional stability analysis of the particle methods where one has to prove that the particles, when evolving in time, stay approximately evenly distributed in space, which usually requires the velocity field to be uniformly Lipschitz continuous (up to a possible logarithmic correction) and rules out the case of L^p initial vorticity fields ([6]). Our analysis takes advantage of the fact that a sequence of random points stay uniformly distributed in a compact manifold as $X = R^2/Z^2$, when advected by a divergence free velocity field, no matter how irregular it is. We believe that our method can be used to prove the convergence of similar particle methods for other non linear problems involving a measure preserving flow, as the Vlasov-Poisson system in Plasma Physics. In [3] we announced a proof of convergence for the genuine vortex method and the Euler equations for non smooth solutions, provided that the vortex particles are randomly distributed at time 0. Unfortunately, the proof (which did not require any rezoning) was not correct. The idea of using random rezoning at each time step turned out to be sufficient to remove the flaw of the proof. Another aspect of this new proof is that, while our primary goal was to study particle approximations to Euler equations, it shows that the numerical dissipation obviously introduced in the rezoning process can be used to mimic the real diffusion in the Navier-Stokes equations.

The paper is organized as follows. In section 2, a general result is stated on random particle approximations in the case of the $d-$ dimensional unit cube with periodic boundary conditions $X = R^d/Z^d$. The main tool is an elementary version of Cramer's theorem in large deviation theory. A straightforward application is a new convergence analysis, in section 3, for particle approximations to the linear (Liouville) equation

$$\partial_t \omega + \mathrm{div}(u\omega) = 0,$$

with divergence free velocity field u. The proof only requires the initial condition $\omega(t=0,\cdot)$ to be a bounded measurable function (or more generally an L^p function, $p>1$) and does not involve any bound on the derivatives of u. In section 4, the convergence of the vortex type method, with random rezoning, is proved for the two dimensional Euler and Navier-Stokes equations, with the help of technical results stated in section 5 (including a rephrasing of standard compactness arguments for non linear evolution equations).

2. Random particle approximations. In the remainder of the paper, X denotes R^d/Z^d, the d-dimensional unit cube with periodic boundary conditions and ζ is a smooth radial compactly supported mollifier on R^d, satisfying $\int \zeta(x)dx = 1$ and for $i,j \in [1,d]$

$$\int_{R^d} x_i x_j \zeta(x)dx = 2\delta_{ij}.$$

We will use the Sobolev spaces $W^{s,r}(X)$, for $s \in R$, $1 \le r \le +\infty$, and denote the corresponding norm by $||\cdot||_{s,r}$ (so that $||\cdot||_{0,r}$ is the usual L^r norm).

We define, for any small $\varepsilon > 0$, an approximation ζ_ε to the delta function on X, by setting, for all $x \in X$,

$$\zeta_\varepsilon(x) = \sum_{i \in Z^d} \varepsilon^{-d} \zeta(\frac{x}{\varepsilon} - i),$$

which enforces

$$\int_X \phi(z)\zeta_\varepsilon(x-z)dz = \int_{R^d} \phi(x+\varepsilon y)\zeta(y)dy,$$

for any integrable function ϕ on X (that is any locally integrable function on R^d, periodic in space).

We also consider the space Ω of all sequences $\xi = (\xi_1, \xi_2, \cdots)$ valued in X. This product space is a probability space for the (countable) product of the Lebesgue measure on each copy of X [17]. Given an L^∞ function ω on X, one can define a smooth space periodic blob approximation

$$\omega_{\varepsilon,N,\xi}(x) = S_\xi^{N,\varepsilon}\omega(x) = \frac{1}{N}\sum_{i=1}^{N} \omega(\xi_i)\zeta_\varepsilon(x-\xi_i),$$

where $\varepsilon > 0$, $N > 0$ and $\xi = (\xi_i) \in \Omega$ are given. Because of the substitution of ζ_ε for the delta function, it is natural to compare the blob approximation with the mollified version of ω defined by $\omega_\varepsilon = \zeta_\varepsilon * \omega$, that is

$$\omega_\varepsilon(x) = \int_{R^d} \omega(x+\varepsilon y)\zeta(y)dy,$$

rather than with ω itself. (The discrepancy between ω and ω_ε will be discussed later - see Proposition 2.1 below.)

THEOREM 2.1. *Let $\omega \in L^\infty(X)$ and $M = ||\omega||_{L^\infty}$. Then, for any $\lambda \in]0, C[$,*

$$||\omega_{\varepsilon,N,\xi} - \omega_\varepsilon||_{0,\infty} \leq \lambda M$$

holds for all $\xi \in \Omega$ outside of a set of measure

$$p_{\varepsilon,N}(\lambda M) \leq C \exp(-CN\varepsilon^d\lambda^2 - d\ \log\lambda - (d+1)\log(\varepsilon^d)), \tag{2.1}$$

where C depends only on X and the mollifier ζ.

THEOREM 2.2. *Let $\omega \in L^\infty(X)$ and $\gamma \in]0, 1/2[$. Assume that*

$$\varepsilon = \varepsilon(N) >> N^{-2\gamma/d}(\log N)^{1/d}$$

when $N \to +\infty$. Then, for almost every $\xi \in \Omega$, there is $N_0 = N_0(\xi)$ such that

$$||\omega_{\varepsilon,N,\xi} - \omega_\varepsilon||_{0,\infty} \leq N^{-1/2+\gamma} \tag{2.2}$$

holds for all $N \geq N_0$.

Proofs of Theorem 2.1 and 2.2. Theorem 2.2 follows from Theorem 2.1 and the Borel-Cantelli lemma. Indeed, because $\varepsilon = \varepsilon(N)$ with $\varepsilon^d >> N^{-2\gamma}\log N$, we have

$$\sum_{N=1}^{\infty} p_{\varepsilon(N),N}(N^{-1/2+\gamma}) \leq C \sum_{N=1}^{\infty} \exp(-s_N)$$

where

$$s_N = CN^{2\gamma}\varepsilon^d + 0(\log\ N) >> \log\ N.$$

Thus, the series converges which implies, by Borel-Cantelli's lemma, that for almost every ξ, inequality (2.2) can fail only for a finite number of values of N.

Let us now prove theorem 2.1 with the help of the following lemma (that can be seen as a very elementary version of the most basic theorem in large deviation theory, namely Cramer's theorem).

LEMMA 2.3. *Let g be a bounded measurable function on X such that*

$$|g(x)| \leq A, \quad \int_X g(x)^2 dx \leq B.$$

Then, for any $\beta \in [0, BA^{-2}]$,

$$|\frac{1}{N}\sum_{i=1}^{N} g(\xi_i) - \int_X g(y)dy| \leq \beta A$$

holds for all ξ outside of a set of measure less than

$$2\exp(-\frac{N\beta^2A^2}{4B}).$$

A proof is given in section 5 for the reader's convenience. Notice that this useful tool is not new in the field of particle method analysis. It has been used by Goodman and Long ([10], [14]) to prove the convergence of the random vortex method for the two and three dimensional Navier-Stokes equations (in the smooth regime).

Let us now prove theorem 2.1 by setting $f(x,y)=\omega(y)\zeta_\varepsilon(x-y)$. Since ζ is smooth, we have the following estimates :

$$|\partial_x f(x,y)| \le CM\varepsilon^{-(d+1)},$$

$$|f(x,y)| \le CM\varepsilon^{-d},$$

and

$$\int_X f(x,y)^2 dy \le CM^2\varepsilon^{-d},$$

where C depends only on ζ. From Lemma 2.1, we deduce that, for each fixed $x \in X$, for any $\beta \in [0,\varepsilon^d]$,

$$|\frac{1}{N}\sum_{i=1}^N f(x,\xi_i) - \int_X f(x,y)dy| \le C\beta M\varepsilon^{-d}$$

holds for all ξ outside of a set of measure less than

$$C\exp(-C\beta^2\varepsilon^{-d}N).$$

By setting $\lambda = C\beta\varepsilon^{-d}$, that is $\beta = C'\lambda\varepsilon^d$, we find that

$$|\frac{1}{N}\sum_{i=1}^N f(x,\xi_i) - \int_X f(x,y)dy| \le \lambda M$$

holds for any $\lambda \in [0,C]$ and any ξ outside of a set of measure bounded by

$$p_{\varepsilon,N}(x,\beta) \le C\exp(-C\lambda^2 N\varepsilon^d).$$

Now, let us introduce a sequence $(x_1,\cdots,x_K)$ of points in X in such a way that, for each $x\in X$, $\mathrm{dist}(x,\{x_1,\cdots,x_K\}) \le h$, where $h>0$ is a given small parameter. Clearly, K can be bounded by Ch^{-d} and therefore

$$\sup_{k=1,\cdots,K} |\frac{1}{N}\sum_{i=1}^N f(x_k,\xi_i) - \int_X f(x_k,y)dy| \le \lambda M$$

holds for all $\lambda \in [0,C]$ and all ξ outside of a set of measure bounded by

$$Ch^{-d}\exp(-C\lambda^2 N\varepsilon^d).$$

Now, for each $x \in X$, we can find $k \in 1,\cdots,K$, such that

$$|\frac{1}{N}\sum_{i=1}^N f(x,\xi_i) - \frac{1}{N}\sum_{i=1}^N f(x_k,\xi_i)| \le CM\ h\varepsilon^{-(d+1)}$$

and

$$|\int_X f(x,y)dy - \int_X f(x_k,y)dy| \le CM\ h\varepsilon^{-(d+1)}.$$

By setting $CM\ h\varepsilon^{-(d+1)} = \lambda M$, that is $h = C\lambda\varepsilon^{d+1}$, we find that, for every $\lambda \in [0, C]$,

$$\sup_{x\in X} |\frac{1}{N}\sum_{i=1}^{N} f(x,\xi_i) - \int_X f(x,y)dy| \leq C\lambda M$$

holds for all ξ outside of a set of measure bounded by

$$C\lambda^{-d}\varepsilon^{-d(d+1)}\exp(-C\lambda^2 N\varepsilon^d).$$

Finally, since $f(x,y) = \omega(y)\zeta_\varepsilon(x-y)$, the proof of Theorem 2.1 is completed.

In Theorem 2.1 and 2.2 the blob approximation was not compared with ω but rather with $\omega_\varepsilon = \zeta_\varepsilon * \omega$. To estimate the discrepancy between ω and ω_ε, we use the negative Sobolev spaces $W^{-s,r}(X)$, and the corresponding norms $||\cdot||_{-s,r}$ for $s \geq 0$ and $1 \leq r \leq +\infty$ (so that $||\cdot||_{0,r}$ is the usual L^r norm).

Proposition 2.4.

$$||\omega_\varepsilon||_{0,r} \leq ||\omega||_{0,r}$$

holds for all $r \in [1,+\infty]$,

$$\lim_{\varepsilon\to 0} ||\omega_\varepsilon - \omega||_{0,r} = 0$$

for all $r \in [1,+\infty[$ *and*

$$||\omega_\varepsilon - \omega - \varepsilon^2\Delta\omega||_{-4,r} \leq C\varepsilon^4,$$

$$||\omega_\varepsilon - \omega||_{-2,r} \leq C\varepsilon^2,$$

where C *depends only on the norm of* ω *in* $L^r(X)$.

The proof is straightforward and is given in section 5 for the reader's convenience.

3. Particle approximations to linear equations. Let $(t,x) \in R\times X \to u(t,x) \in R^d$ be a given Lipschitz continuous divergence free velocity field on X and denote by $\exp(t_0,t_1,u)y_0$ the solution at time t_1 of the ODE $y' = u(t,y)$ with initial condition $y(t_0) = y_0$. Then, for each pair (t_0,t_1),

$$y \in X \to \exp(t_0,t_1,u)y$$

defines one-to-one mapping g from X to itself, Lipschitz continuous in both directions, that preserves the Lebesgue measure in the sense that

$$\int_X \phi(g(x))dx = \int_X \phi(x)dx$$

holds for any Lebesgue integrable function ϕ on X. The exact solution to the Liouville equation

$$\partial_t\omega + \mathrm{div}(u\omega) = 0,$$

with initial condition

$$\omega(t=0,\cdot) = \omega^0 \in L^\infty(X),$$

is given by the method of characteristics

$$\omega(t,x) = \omega^0(\exp(t,0,u)x)$$

for all $t \in R$ and $x \in X$. The standard particle solution to this problem is given by

$$\omega_{\varepsilon,N,\xi}(t,x) = \frac{1}{N}\sum_{i=1}^{N} \omega^0(\xi_i)\zeta_\varepsilon(x - \exp(0,t,u)\xi_i),$$

where $\xi_1, \cdots, \xi_N$ are the locations of the particles at time 0. When ξ is random, we can prove almost sure convergence as N goes to ∞, exactly in the same way as in section 2. Indeed, the key estimate of theorem 2.1 is in no way affected by the fact that particles have been moved from time 0 to time t by a measure preserving flow. Thus, we have

THEOREM 3.1. *Let $\omega^0 \in L^\infty(X)$, $t > 0$. Assume that $\varepsilon = \varepsilon(N)$ so that*

$$1 >> \varepsilon^d >> N^{-1+\theta}$$

holds for some $\theta > 0$. Then, for almost every $\xi \in \Omega$,

$$\lim_{N\to\infty} ||\omega_{\varepsilon,N,\xi}(t,\cdot) - \omega(t,\cdot)||_{0,r} = 0,$$

holds for all $1 \leq r < +\infty$.

Proof. Let us denote by M the L^∞ norm of the initial data ω^0 and introduce the mollified solution at time t (where t is fixed)

$$\omega_\varepsilon(t,x) = \int_X \omega^0(\exp(t,0,u)z)\zeta_\varepsilon(x-z)dz,$$

or, by using the measure preserving change of variable $z = \exp(0,t,u)y$,

$$\omega_\varepsilon(t,x) = \int_X \omega^0(y)\zeta_\varepsilon(x - \exp(0,t,u)y)dz.$$

On the one hand, we have

$$||\omega_\varepsilon(t,\cdot) - \omega(t,\cdot)||_{0,r} = ||\zeta_\varepsilon * \omega(t,\cdot) - \omega(t,\cdot)||_{0,r} \to 0,$$

when $\varepsilon \to 0$, for all $1 \leq r < +\infty$. On the other hand we have, exactly as in Theorem 2.1,

$$||\omega_{\varepsilon,N,\xi}(t,\cdot) - \omega_\varepsilon(t,\cdot)||_{0,\infty} \leq \lambda M \tag{3.1}$$

for all $\lambda \in]0, C[$ and all $\xi \in \Omega$ with probabilty larger than

$$1 - C\exp(-CN\varepsilon^d\lambda^2 - d\,\log\lambda - (d+1)\log(\varepsilon^d)).$$

Indeed, $\exp(0,t,u)$ induces an obvious $\Omega-$measure preserving mapping defined by

$$(\xi_1, \xi_2, ..) \in \Omega \to (\exp(0,t,u)\xi_1, \cdots) \in \Omega,$$

and therefore the probability that (3.1) fails is exactly the same as in Theorem 2.1. Thus, if ε is chosen as in Theorem 2.2, then

$$||\omega_{\varepsilon,N,\xi}(t,\cdot) - \omega_\varepsilon(t,\cdot)||_{L^\infty} \to 0$$

for almost every sequence ξ, which completes the proof. □

This result can be easily extended when ω^0 is given in $L^p(X)$, $1 < p < +\infty$. In this case we introduce an extra cutoff parameter $R = R(N) = CN^\delta$, for some C and $\delta > 0$ and modify the blob approximation by setting

$$\omega_{\varepsilon,N,R,\xi}(t,x) = \sum_{i=1}^{N} \omega_R^0(\xi_i)\zeta_\varepsilon(x - \exp(0,t,u)\xi_i),$$

where

$$\omega_R^0(x) = \max(-R, \min(R, \omega^0(x))).$$

Then a slight modification of the previous argument shows that the approximation converges, in the L^r sense for all $1 \leq r < p$, and for almost every sequence ξ. (Substitute $R = CN^\delta$ for M in the former estimates and observe that

$$||\omega_R^0 - \omega^0||_{0,r} \leq R^{1-p/r}||\omega^0||_{L^p}^{p/r},$$

by Markov's inequality.)

Remark The proof can also be extended to the case when the velocity field is not Lipschitz continuous (in x) but only belongs to the Sobolev space $W^{1,1}(X)$ (in x). Then, the Liouville equation has to be considered in the sense of DiPerna and Lions (see [7]). The flow map $\exp(0,t,u)$ is a well defined (in the almost everywhere sense) one-to-one measure preserving mapping from X and into itself. Thus the particle method is well defined for almost every sequence ξ (not necessarily all of them !) and the previous proof needs no modification.

4. Particle approximations to two dimensional Euler and Navier-Stokes equations.

Definition of the particle method. Let us introduce some notations. The viscosity of the fluid, ν, is either 0 (Euler equations) or > 0 (Navier-Stokes equations.) The initial vorticity field is denoted by ω^0 and supposed to belong to $L^p(X)$ for some $p > 4/3$. Since the problem is periodic in space, ω^0 is supposed to have zero mean. The final time $T > 0$ is fixed. Let us introduce a small parameter $h > 0$ which will measure the truncation error of the scheme. To simplify the notations, all discretization and cut-off parameters, as well as the number of particles, will be related to h, in the following way :

$$\Delta t = h^\alpha,$$

$$\varepsilon = \begin{cases} h^\beta & \text{if } \nu = 0 \\ \sqrt{\nu} h^\beta & otherwise, \end{cases}$$

$$R^{-1} = h^\delta$$

and, to account for the fact that the particle discretization must be finer than the other discretization parameters we will write

$$N(\log N)^{-1} >> h^{-\gamma}. \tag{4.1}$$

In the case of Navier-Stokes equations ($\nu > 0$), Δt and ε are linked by the additional law

$$\varepsilon^2 = \nu \Delta t,$$

which requires $\alpha = 2\beta$. The positive exponents α, β, γ and δ will be defined later.

As in section 2, the blob approximation of a given function w is denoted by

$$S_\xi^{N,\varepsilon} w(x) = \frac{1}{N} \sum_{i=1}^{N} w(\xi_i) \zeta_\varepsilon (x - \xi_i).$$

Let us also denote by $\tau_k(\xi)$ the shifted sequence

$$\tau_k(\xi) = (\xi_{k+1}, \xi_{k+2}, \cdots),$$

for any positive integer k and any sequence $\xi \in \Omega$. We can now precisely define the particle scheme as follows

1) For $n = 0$, we set

$$\omega_0(x) = \max(-R/2, \min(R/2, \omega^0(x))), \quad \forall x \in X.$$

2) For $n = 0, 1, 2, \cdots$, we get

$$S_{\tau_{nN}(\xi)}^{N,\varepsilon} \omega_n \;\; = \;\; \frac{1}{N} \sum_{i=nN+1}^{(n+1)N} \omega_n(\xi_i) \zeta_\varepsilon (x - \xi_i)$$

and denote by $m_{n+1/2}$ its mean. Notice that

$$m_{n+1/2} = \frac{1}{N} \sum_{i=nN+1}^{(n+1)N} \omega_n(\xi_i).$$

3) We substract out the mean and set

$$\omega_{n+1/2} = S_{\tau_{nN}(\xi)}^{N,\varepsilon} \omega_n - m_{n+1/2}.$$

4) We solve the (2 dimensional periodic) Poisson problem

$$\mathrm{div} u = 0, \quad \mathrm{curl} u = \omega_{n+1/2}$$

and set $u_{n+1/2} = u$.

5) We advect the vorticity field by the time independent velocity field $u_{n+1/2}$ from $t = n\Delta t$ to $t = (n+1)\Delta t$, by solving

$$\partial_t \omega + \mathrm{div}(\omega u_{n+1/2}) = 0,$$

with initial condition $\omega(n\Delta t) = \omega_{n+1/2}$. Then we set $\omega_{n+1} = \omega((n+1)\Delta t)$.

Notice that the sequence (ω_n) depends on h through Δt, ε, R and N, as well as on $\xi \in \Omega$. However, for each n, ω_n and $\omega_{n+1/2}$ depend only on the $N(n+1)$ first terms of ξ.

Let us check that the scheme is well defined. Assume that ω_n is smooth (which is true for $n = 0$) and prove that $\omega_{n+1/2}$ and ω_{n+1} are also smooth. By definition, $\omega_{n+1/2}$

is as smooth as ζ_ε. Then, by standard elliptic theory, $u_{n+1/2}$ is also smooth. Thus, ω_{n+1} is uniquely defined and satisfies

$$||\omega_{n+1}||_{0,r} = ||\omega_{n+1/2}||_{0,r}$$

for all $r \in [1, +\infty]$. Notice that the advection step preserves the mean. Thus, since, by definition, $\omega_{n+1/2}$ has zero mean, the same property holds for ω_{n+1}. So, the scheme is well defined. Let us now show that it corresponds to the particle method described in the introduction (which is not obvious at the first glance). To do that, let us start from step $n-1/2$. $\omega_{n-1/2}$ can be seen as a blob approximation to the vorticity field

$$\omega_{n-1/2}(x) = \frac{1}{N}\sum_{i=1}^{N} \alpha_{n,i}[\zeta_\varepsilon(x - x_{n,i}) - 1],$$

where $x_{n,i}$ and $\alpha_{n,i}$ denote the center of mass and the weight of the blobs for $i = 1, \cdots, N$. The velocity field $u = u_{n-1/2}$ is obtained from $\omega_{n-1/2}$ by solving the Poisson system $\mathrm{div} u = 0$, $\mathrm{curl} u = \omega_{n-1/2}$. Then $\omega_{n-1/2}$ is advected by $u_{n-1/2}$ and the result is

$$\omega_n(x) = \omega_{n-1/2}(\exp(\Delta t, 0, u_{n-1/2})(x)),$$

where $\exp(\cdot, \cdot, u_{n-1/2})$ is the measure preserving flow on X generated by $u_{n-1/2}$. Next, $\omega_{n+1/2}$ is obtained by resampling ω_n on points $\xi_{nN+1}, \cdots, \xi_{(n+1)N}$, which means :

$$\omega_{n+1/2}(x) = \frac{1}{N}\sum_{i=1}^{N} \alpha_{n+1,i}[\zeta_\varepsilon(x - x_{n+1,i}) - 1],$$

where $x_{n+1,i} = \xi_{nN+i}$, for $i = 1, \cdots, N$ and

$$\alpha_{n+1,i} = \omega_n(\xi_{nN+i}) = \omega_{n-1/2}(\exp(\Delta t, 0, u_{n-1/2})(\xi_{nN+i})).$$

Now, if we set

$$x_{n+1/2,i} = \exp(\Delta t, 0, u_{n-1/2})(\xi_{nN+i}),$$

or, equivalently,

$$\xi_{nN+i} = \exp(0, \Delta t, u_{n-1/2})(x_{n+1/2,i}),$$

we get

$$\omega_{n+1/2}(x) = \frac{1}{N}\sum_{i=1}^{N} \alpha_{n+1,i}[\zeta_\varepsilon(x - x_{n+1,i}) - 1],$$

where

$$x_{n+1,i} = \exp(0, \Delta t, u_{n-1/2})(x_{n+1/2,i})$$

and

$$\alpha_{n+1,i} = \omega_{n-1/2}(x_{n+1/2,i}).$$

Notice that the sequence $(x_{n+1/2,i}, i = 1, \cdots, N)$ is obtained from the random sequence $(\xi_{nN+i}, i = 1, \cdots, N)$ by a measure preserving mapping $x \to \exp(\Delta t, 0, u_{n-1/2})$, and, therefore, can also be considered as a random sequence on X, independent of the ξ_j for $j \leq nN$. Thus, if we use the change of notation $\Omega_n = \omega_{n-1/2}$, we exactly recover the particle scheme described in the introduction.

Observe that, besides being a crucial ingredient in the proof of the stability, the random rezoning allows also to reduce the convection of the particles to the explicit resolution of the advection equation for the vortex blob approximation. This will greatly simplify the proof of the consistency of the scheme.

Convergence of the particle method. In a standard way, the approximate solution can be defined for all $x \in X$ and all $t \in [0, T]$ by setting

$$\omega^{h,\xi}(t,x) = \omega_n(x) + (\frac{t}{\Delta t} - n)(\omega_{n+1}(x) - \omega_n(x)),$$

when $n\Delta t \leq t \leq (n+1)\Delta t$, $n = 0, 1, 2, \cdots$

Depending on how the discretization parameters are scaled, we get two different convergence results.

THEOREM 4.1.

Assume that $\nu = 0$, $\alpha = 1 + \delta\eta$, $\beta = 1 + \delta\eta/2$, $\gamma = 6 + \delta(2 + 3\eta)$, *where* $\delta > 0$, $\eta = 0$ *if* $p > 2$ *and* $\eta > 3(1 - p/2)$ *if* $4/3 < p \leq 2$.
Then, for almost every $\xi \in \Omega$, *there is a sequence* $h_k \to 0$ *such that the corresponding approximate solution* $\omega^{h_k,\xi}$ *converges weakly to* ω , *where* ω *is a weak solution to the initial value problem for the Euler equation.*

THEOREM 4.2.

Assume now that $\nu > 0$, $\alpha = 1 + \delta\eta$, $\beta = 1/2 + \delta\eta/2$, $\gamma = 5 + \delta(2 + 3/2\eta)$.
Then, the same result holds, except that, now, ω *solves the Navier-Stokes equation*

$$\partial_t\omega + div(\omega\, curl(-\Delta)^{-1}\omega) = \nu\Delta\omega$$

in the sense of distribution.

Remark

i) We will actually prove that, for a subsequence, the convergence of ω^h to ω is in the $L^\infty(0, T; L^p(X))$ weak-* sense, and is strong in $C^0(0, T; W^{-1/2,2}(X))$

ii) Notice that the distribution valued quadratic operator

$$A(\omega) = \mathrm{div}(\omega \mathrm{curl}(-\Delta)^{-1}\omega),$$

is well defined and continuous on the negative Sobolev space $W^{-1/2,2}(X)$ (since $(-\Delta)^{-1}$ maps $W^{-1/2,2}$ into $W^{3/2,2}$) with values in $\mathrm{div}(L^1(X))$. By Sobolev imbeddings, if $p > 4/3$ it is therefore continuous, for instance, from $L^p(X)$ into $W^{-3,p}(X)$.

The proof is based on the following preliminary estimates.

PROPOSITION 4.3.

For almost every $\xi \in \Omega$, *there is* $N_0 = N_0(\xi, T, \omega_0)$ *such that*

$$||S^{N,\varepsilon}_{\tau_{nN}(\xi)}\omega_n - \zeta_\varepsilon * \omega_n||_{0,\infty} \leq \frac{1}{2}h\Delta t \tag{4.2}$$

$$||\omega_{n+1/2} - \zeta_\varepsilon * \omega_n||_{0,\infty} \leq h\Delta t \tag{4.3}$$

and

$$(4.4) \qquad ||\omega_{n+1/2}||_{0,\infty} \le R$$

hold for all integer n such that $n\Delta t \le T$ and all $N \ge N_0$.

Proof. Notice first that (4.3) immediately follows from (4.2). Indeed, since ω_n has zero mean, the same holds for $\zeta_\varepsilon * \omega_n$, thus, we have

$$|m_{n+1/2}| = |\int_X [S^{N,\varepsilon}_{\tau_n N(\xi)}\omega_n(x) - \zeta_\varepsilon * \omega_n(x)]dx| \le \frac{1}{2}h\Delta t$$

(by (4.2)), which implies (4.3), by definition of $\omega_{n+1/2}$.
To prove (4.2), we use Theorem 2.1, by considering a collection of functions (ω_n, $1 \le n \le T/\Delta t$), uniformly bounded by R, rather than a single function bounded by M as in Theorem 2.2. Let us assume that (4.4) is already proved. For fixed values of h, N, ε, R and Δt, the probability p that (4.2) fails for at least one value of n in the range $[0, T/\Delta t]$ can be bounded in view of (2.1), with $d = 2$, $\lambda = h\Delta t/R$ and $M = R$. The power laws giving all discretization parameters in terms of h allow to bound all their logarithmic contributions by $C \log N$ which yields

$$p \le CT/\Delta t \exp(-CN\varepsilon^d(h\Delta t/R)^2 + C\log\ N).$$

Next, since by (2.4) we have $\Delta t = h^\alpha \ge N^{-\alpha/\gamma}$, we get

$$p \le C\exp(-CN(\varepsilon h\Delta t/R)^2 - C'\log N).$$

Now let us check how the exponential decay in N is able to absorb any polynomial contributions (as the one due to the addition of $0(1/\Delta t)$ 'bad' events). This probability, written as a function of N, is the general term of a convergent series as soon as

$$N(\varepsilon h\Delta t/R)^2 >> \log N.$$

This certainly occurs since $\Delta t = h^\alpha$, $\varepsilon = h^\beta$, $R^{-1} = h^\delta$, $N(\log N)^{-1} >> h^{-\gamma}$ and

$$\alpha + \beta + \delta + 1 \le \gamma/2$$

is satisfied by our choice of α, β, δ and γ. Then, by Borel-Cantelli lemma (as for the proof of Theorem 2.2), we conclude that (4.2) holds for almost every $\xi \in \Omega$, provided that N is larger than $N_0 = N_0(\xi)$.

Notice that it is essential, when defining $\omega_{n+1/2}$ from ω_n, to use 'fresh random numbers', namely the entries of ξ from rank $nN + 1$ up to $(n+1)N$. Otherwise, the estimate would break down. (It is for a related reason that the proof of [3] is not correct.)
Let us now prove (4.4). (As a matter of fact, (4.4) should be proved together with (4.2), since (4.2) requires (4.4), but we skip the obvious induction argument to shorten the proof.) By assumption, $||\omega_0||_{0,\infty}$ is bounded by $R/2$. Moreover, at each time step, the L^∞ norm of $\zeta_\varepsilon * \omega_n$ is not larger that the one of ω_n, and the advection step preserves the L^∞ bound. Thus (4.4) follows from (4.2),(4.3), which achieves the proof of Proposition 4.3. ☐

From these basic estimates, we deduce

PROPOSITION 4.4.

Let us denote by $C > 0$ various constants depending on the initial condition ω^0 and by A the quadratic operator

$$A(\omega) = div(\omega curl(-\Delta)^{-1}\omega).$$

Then, for almost every $\xi \in \Omega$, there is $N_0 = N_0(\xi)$ such that

$$||\omega_n||_{0,p} \le C, \quad ||\omega_{n+1/2}||_{0,p} \le C, \tag{4.5}$$

$$||\frac{1}{\Delta t}(\omega_{n+1} - \omega_{n+1/2}) + A(\omega_{n+1/2})||_{-4,p} + ||\frac{1}{\Delta t}(\omega_{n+1/2} - \omega_n) + \nu\Delta\omega_n||_{-4,p} \le Ch \tag{4.6}$$

for all integer n such that $n\Delta t \le T$ and all $N \ge N_0$.

Proof. Since the advection operator preserves the L^p norm, we have

$$||\omega_{n+1}||_{0,p} = ||\omega_{n+1/2}||_{0,p}.$$

We also have

$$||\omega_{n+1/2}||_{0,p} \le ||\zeta_\varepsilon * \omega_n||_{0,p} + h\Delta t$$

(because of (4.2))

$$\le ||\omega_n||_{0,p} + h\Delta t.$$

Thus, (4.5) immediately follows. Let us now prove (4.6). On the one hand, we have

$$||\omega_{n+1/2} - \omega_n - \nu\Delta t\Delta\omega_n||_{-k,p} \le ||\omega_{n+1/2} - \zeta_\varepsilon * \omega_n||_{0,p}$$

$$+||\zeta_\varepsilon * \omega_n - \omega_n - \nu\Delta t\Delta\omega_n||_{-k,p} \le h\Delta t + C'\varepsilon^k,$$

(because of (4.2) and Proposition 2.1), with $k = 2$ in the Euler case ($\nu = 0$) and $k = 4$, $C' = C\nu$, $\varepsilon^2 = \nu\Delta t$, in the Navier-Stokes case ($\nu > 0$). On the other hand, the following consistency result can be proved for the 'advection step' that transforms $\omega_{n+1/2}$ into ω_{n+1} (see Proposition 5.1 in section 5)

$$||\omega_{n+1} - \omega_{n+1/2} + \Delta t A(\omega_{n+1/2})||_{-k,p} \le C_\eta R^\eta \Delta t^2,$$

where $\eta \ge 0$ depends on p. (More precisely $\eta = 0$ if $p > 2$ and $\eta > 3(1 - p/2)$ if $4/3 < p \le 2$.) So, (4.6) follows as soon as Δt, R and ε satisfy :
a) in the case $\nu = 0$,

$$\varepsilon^2 + R^\eta \Delta t^2 \le Ch\Delta t,$$

that is $2\beta \ge 1 + \alpha$ and $-\eta\delta + \alpha \ge 1$.
b) in the case $\nu > 0$ (where $\alpha = 2\beta$),

$$\nu\varepsilon^4 + R^\eta \Delta t^2 \le Ch\Delta t,$$

that is $4\beta \ge 1 + \alpha$ and $-\eta\delta + \alpha \ge 1$.
In both cases, this is automatically enforced with our choice of α, β, γ and δ. The proof of Proposition 4.4 is now completed. $\square$

The convergence theorems then follow through classical compactness arguments. (see [13], for instance or more specifically [12]) that we now sketch. Let us fix ξ so that (4.5) and (4.6) hold and use

$$\omega^h(t,x) = \omega_n(x) + (\frac{t}{\Delta t} - n)(\omega_{n+1}(x) - \omega_n(x)),$$

when $n\Delta t \le t \le (n+1)\Delta t$, $n \ge 0$. As already observed, A is continuous form $L^p(X)$ into $W^{-3,p}(X)$. Since Δ is clearly continuous from $L^p(X)$ into $W^{-2,p}(X)$, it follows from (4.5) and (4.6) that ω^h and $\partial_t \omega^h$ are respectively bounded in $L^\infty(0,T;L^p(X))$ and $L^\infty(0,T;W^{-4,p}(X))$.
The space $L^p(X)$ is compactly embedded in $W^{-1/2,2}(X)$, which is continuously embedded in $W^{-4,p}(X)$. Therefore

$$(\omega^h)_h \text{ is relatively compact in } C^0(0,T;W^{-1/2,2}(X)) \ .$$

Thus, there is a sequence $h_k \to 0$ and a function $\omega \in L^\infty(0,T;L^p(X))$ such that ω^{h_k} converge toward ω in $C^0(0,T;W^{-1/2,2}(X))$. In particular, $\omega(t=0,.) = \omega_0$. Finally, because of the continuity of A and Δ, it follows from (4.6) that ω is a solution to the differential equation

$$\partial_t \omega + A(\omega) + \nu\Delta\omega = 0,$$

which completes this abridged proof.
Remark We have chosen here to emphasize the role of the truncation error h, which leads to a rather severe constraint on the number of particles. For instance, when $\nu = 0$, $p = \infty$ (and, then, $\eta = 0$), N must satisfy $N >> h^{-6}$, meanwhile Δt and ε are $0(h)$. However, since the convergence theorem does not state any error estimate, one might prefer to stress the optimal relationship between the cut-off parameter and the initial average spacing between particles. Then, by allowing the truncation error in Proposition 4.1 to be of order h^ρ with $\rho > 0$, arbitrarily small, it is easy to check that convergence holds as soon as $N >> \varepsilon^{-4}$, which must be compared to the condition $N >> \varepsilon^{-2}$ classically obtained in the smooth case. It is also worthy to recall that in absence of random rezoning this condition woud become of the type $N >> \exp(CT/\varepsilon)$

5. Technical results.

Proof of Lemma 2.1. Let us introduce

$$\gamma(x) = g(x) - \int_X g(y)dy$$

and estimate, for any fixed $\beta > 0$, the measure $p_+(\beta)$ of the ξ such that

$$\frac{1}{N}\sum_{i=1}^N \gamma(\xi_i) \ge \beta A.$$

We have, for any $\alpha > 0$,

$$p_+(\beta)\exp(\alpha\beta A) \le \int_{X^N} \exp(\alpha(\frac{1}{N}\sum_{i=1}^N \gamma(\xi_i)))d\xi$$

$$= (\int_X \exp(\frac{\alpha}{N}\gamma(y)dy)^N.$$

Since $\exp(t) \leq 1 + t + t^2$ for $|t| \leq 1$, we get

$$\exp(\frac{\alpha}{N}\gamma(y)) \leq 1 + \frac{\alpha}{N}\gamma(y) + (\frac{\alpha}{N}\gamma(y))^2,$$

as soon as $2\alpha A \leq N$. Thus,

$$p_+(\beta)\exp(\alpha\beta A) \leq (\int_X (1 + \frac{\alpha}{N}\gamma(y) + (\frac{\alpha}{N}\gamma(y))^2)dy)^N$$

$$= (1 + \frac{\alpha^2}{N^2}\int_X \gamma(y)^2 dy)^N$$

(since the integral of γ is zero)

$$\leq \exp(\frac{\alpha^2 B}{N}).$$

So, we have

$$p_+(\beta) \leq \inf_{\alpha \in]0,N/(2A)[} \exp(-\alpha\beta A + \frac{\alpha^2 B}{N})$$

$$= \exp(-\frac{N\beta^2 A^2}{4B})$$

when $\beta \leq BA^{-2}$. Since the measure $p_-(\beta)$ of the ξ such that

$$\frac{1}{N}\sum_{i=1}^{N}\gamma(\xi_i) \leq \beta A$$

can be estimated in the same way, this achieves the proof.

Proof of Proposition 2.1. The proof follows from simple Taylor expansions. For any smooth test function ϕ, we have :

$$\int_X [\omega_\varepsilon(x) - \omega(x)]\phi(x)dx = \int_X \int_{R^d} \omega(x)[\phi(x - \varepsilon y) - \phi(x)]\zeta(y)dydx$$

$$= \int_X \int_{R^d} \omega(x)[-\varepsilon D\phi(x)y + \frac{\varepsilon^2}{2}D^2\phi(x)(y,y) - \frac{\varepsilon^3}{6}D^3\phi(x)(y,y,y)]\zeta(y)dydx + r$$

$$= \varepsilon^2 \int_X \omega(x)\Delta\phi(x)dx + r,$$

where

$$|r| \leq C\varepsilon^4 ||D^4\phi||_{0,p'}||\omega||_{0,p},$$

C depends only on ζ and $p' = (1 - p^{-1})^{-1}$. It follows that

$$||\omega_\varepsilon - \omega - \varepsilon^2\Delta\omega||_{-4,p} \leq C\varepsilon^4.$$

In the same way, we get

$$||\omega_\varepsilon - \omega||_{-4,p} \leq C\varepsilon^2.$$

5.1. Technical results on the advection step. Let us first state the

PROPOSITION 5.1. *Let ω_0 be a smooth function on X with zero mean, $u_0 = curl(-\Delta)^{-1}\omega_0$. Then there is a unique smooth solution $\omega(t,x)$ of the initial value problem*

$$\partial_t\omega + div(u_0\omega) = 0,$$

$$\omega(0,x) = \omega_0(x).$$

This solution satisfies the following properties for all positive t :

$$||\omega(t)||_{0,q} = ||\omega_0||_{0,q}, \tag{5.1}$$

for all $q \in [1,+\infty]$ and

$$\int_X \omega(t,x)dx = \int_X \omega_0(x)dx = 0. \tag{5.2}$$

Moreover, for $4/3 < p < +\infty$, there are various constants C, depending only on the norm of ω_0 in $L^p(X)$ and there is $\sigma > 0$ such that, for any positive t,

$$||\partial_t\omega(t)||_{-1,1+\sigma} \leq C, \tag{5.3}$$

$$||\partial_{tt}\omega(t)||_{-2,1+\sigma} \leq C_\eta R^\eta, \tag{5.4}$$

$$||\omega(t) - \omega_0 + t\,div(u_0\omega_0)||_{-2,1+\sigma} \leq C_\eta R^\eta t^2,$$

where

$$R = ||\omega_0||_{0,\infty}, \tag{5.5}$$

$\eta = 0$ if $p > 2$ and $\eta > 3(1-p/2)$ otherwise.

Proof. In the sequel, any constant will be denoted by C and may depend on the norm of ω_0 in $L^p(X)$.
Since ω_0 is smooth, the initial value problem has a unique smooth solution which satisfies

$$||\omega(t)||_{0,q} = ||\omega_0||_{0,q},$$

for all q. This proves (5.1). Property (5.2) readily results from the equation satisfied by ω.
We now turn to (5.3). We have

$$\partial_t\omega = -\mathrm{div}(u_0\omega).$$

Thus

$$||\partial_t\omega(t)||_{-1,1+\sigma} \leq ||u_0\omega(t)||_{0,1+\sigma}, \tag{5.6}$$

for any positive σ. By Hölder's inequality, we obtain

$$||u_0\omega(t)||_{0,1+\sigma} \leq ||u_0||_{0,q}||\omega(t)||_{0,p} \leq C||u_0||_{0,q}, \tag{5.7}$$

where C depends only on the L^p norm of ω_0 and q is such that $q^{-1}+p^{-1}=1+\sigma^{-1}$. Next, since $u_0 = \mathrm{curl}(-\Delta)^{-1}\omega_0$, by Sobolev's embedding theorem, we get

$$||u_0||_{0,q} \leq C,$$

where C depends only on the L^p norm of ω_0, as soon as $q^{-1} > (p^{-1}-2^{-1})_+$. But $p \geq 4/3$ and therefore $2/p - 1/2 > 1$, so that it is possible to choose $\sigma > 0$ such that $q^{-1} > (p^{-1}-2^{-1})_+$. By Sobolev's embedding theorem, we thus get

$$||u_0||_{0,q} \leq C,$$

where C depends only on the L^p norm of ω_0, which combined with (5.6) and (5.7) yields (5.3). The last estimate is obtained in a more complicated way. We write

$$\partial_{tt}\omega = -\partial_{jt}(u_{0j}\omega)$$

(with the usual summation convention),

$$= -\partial_j(u_{0j}\partial_t\omega)$$

(since u_0 is time independent),

$$= \partial_j(u_{0j}\partial_i[u_{0i}\omega])$$

$$= \partial_{ji}(u_{0j}u_{0i}\omega) - \partial_j(u_{0i}\omega\partial_i u_{0j}).$$

Thus, for any positive σ,

$$||\partial_{tt}\omega(t)||_{-2,1+\sigma} \leq ||u_0\,\omega(t)\;u_0||_{0,1+\sigma} + ||Du_0\,\omega(t)\;u_0||_{0,1+\sigma}.$$

Let us estimate only the second term of the right hand side (the first one being simpler).
Since $u_0 = \mathrm{curl}(-\Delta)^{-1}\omega_0$, by Sobolev's embedding theorem, we have

$$||u_0||_{0,q} \leq C,$$

for any q such that $q^{-1} > (p^{-1}-2^{-1})_+$, where C depends only on q and the L^p norm of ω_0. Thus

$$||Du_0\,\omega(t)\;u_0||_{0,1+\sigma} \leq C||Du_0\,\omega(t)||_{0,r},$$

with r such that $r^{-1}+q^{-1}=1+\sigma^{-1}$. By choosing σ small enough, this allows to pick any r satisfying $r^{-1} < 1-(p^{-1}-2^{-1})_+$ (q and σ being then chosen as functions of r). By Schwarz inequality, we have

$$||Du_0\,\omega(t)||_{0,r} \leq ||Du_0||_{0,2r}||\omega(t)||_{0,2r}.$$

Since, by (5.1),

$$||\omega(t)||_{0,2r} \leq ||\omega_0||_{0,2r}$$

and

$$||Du_0||_{0,2r} \leq ||\omega_0||_{0,2r}$$

(because $u_0 = \mathrm{curl}(-\Delta)^{-1}\omega_0$), we deduce

$$||Du_0\, \omega(t)||_{0,r} \le ||\omega_0||_{0,2r}^2.$$

When $p > 2$, we can take any r in $]1, +\infty[$ and thus set $2r = p$. Then

$$||Du_0\, \omega(t)||_{0,r} \le C,$$

where C depends only on the L^p norm of ω_0. When $4/3 < p \le 2$, we must satisfy $2r > p$ and the L^∞ bound R on ω_0 has to be used. By Hölder's inequality, we get

$$||\omega_0||_{0,2r} \le CR^{1-p/(2r)},$$

and, thus,

$$||Du_0\, \omega(t)||_{0,r} \le CR^{2-p/r}.$$

From these estimates, we deduce

$$||Du_0\, \omega(t)\ u_0||_{0,1+\sigma} \le C_\eta R^\eta,$$

with $\eta = 0$ if $p > 2$ and $\eta > 3(1-p/2)$, provided that $\sigma > 0$ is sufficiently small. This gives (5.4) from which (5.5) easily follows. □

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