Laplace and Poisson equations: physical modelling

Many physical phenomena are governed by diffusion (heat distribution, chemical or biological concentration, electric potential...)

**Steady state solution:** for any region $\omega \subset \Omega$, balance between

- fluxes through the boundary $\partial \omega$
- sinks/sources within $\omega$

$$\int_{\partial \omega} \Phi \cdot n \, ds = \int_{\omega} f(x) \, dx$$
Laplace and Poisson equations: physical modelling

Many physical phenomena are governed by **diffusion** (heat distribution, chemical or biological concentration, electric potential...)

**Steady state solution:** for any region $\omega \subset \Omega$, balance between

- fluxes through the boundary $\partial \omega$
- sinks/sources within $\omega$

$$\int_{\partial \omega} \Phi \cdot n \, ds = \int_{\omega} f(x) \, dx \quad \implies \quad \text{div} \, \Phi(x) = f(x) \quad \forall x \in \Omega$$
Laplace and Poisson equations: physical modelling

Many physical phenomena are governed by diffusion (heat distribution, chemical or biological concentration, electric potential...)

Steady state solution: for any region $\omega \subset \Omega$, balance between

- fluxes through the boundary $\partial \omega$
- sinks/sources within $\omega$

$$\int_{\partial \omega} \Phi \cdot n \, ds = \int_{\omega} f(x) \, dx \quad \implies \quad \text{div} \, \Phi(x) = f(x) \quad \forall x \in \Omega$$

A common law: (Fourier’s law, Fick’s law) $\Phi = -k \nabla u$ where $u$ is the density (temperature, concentration...) and $k$ is the diffusivity coefficient

$$-\text{div} \, (k(x) \nabla u(x)) = f(x)$$
Laplace and Poisson equations

\[-\text{div} \left( k(x) \nabla u(x) \right) = f(x)\]

- If \( k \) depends on \( u \): nonlinear equation
- If \( k \) does not depend on \( u \): linear equation
- If \( k \) is a constant: \(-\Delta u = \frac{1}{k} f\) Poisson equation
- If \( k \) is a constant and \( f = 0 \): \( \Delta u = 0 \) Laplace equation, harmonic functions
Some harmonic functions in $\mathbb{R}^2$

- Polynomials:

  \[ u(x, y) = a(x^2 - y^2) + bxy + cx + dy + e \]

- Separation of variables in cartesian coordinates:

  \[ u(x, y) = v(x)w(y) \]

  Elementary solutions:

  \[ u_{\lambda}(x, y) = \begin{cases} 
    (a \cos \lambda x + b \sin \lambda x)(c e^{\lambda y} + d e^{-\lambda y}) \\
    (a e^{\lambda x} + b e^{-\lambda x})(c \cos \lambda y + d \sin \lambda y)
  \end{cases} \forall \lambda \neq 0 \]

  Hence any convergent sum of these functions
Some harmonic functions in $\mathbb{R}^2$

- Polynomials:
  \[ u(x, y) = a(x^2 - y^2) + bxy + cx + dy + e \]

- Separation of variables in cartesian coordinates
  \[ u(x, y) = v(x)w(y) \]
Some harmonic functions in \( \mathbb{R}^2 \)

- **Polynomials:**
  
  \[
  u(x, y) = a(x^2 - y^2) + bxy + cx + dy + e
  \]

- **Separation of variables in cartesian coordinates**
  \[
  u(x, y) = v(x)w(y)
  \]

  elementary solutions: \( u_\lambda(x, y) = \begin{cases} (a \cos \lambda x + b \sin \lambda x)(ce^{\lambda y} + de^{\lambda y}) & \forall \lambda \neq 0 \\ (ae^{\lambda x} + be^{\lambda x})(c \cos \lambda y + d \sin \lambda y) \end{cases} \)

  hence any convergent sum of these functions
Some harmonic functions in $\mathbb{R}^2$

- Separation of variables in polar coordinates $u(r, \theta) = \nu(r)w(\theta)$

$$\Delta u(r, \theta) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}$$
Some harmonic functions in $\mathbb{R}^2$

- Separation of variables in polar coordinates $u(r, \theta) = v(r)w(\theta)$

$$\Delta u(r, \theta) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}$$

Elementary solutions on $\mathbb{R}^2 \setminus (0,0)$

$$\begin{cases} 
  u_0(r, \theta) = c_0 \ln r + d_0 \\
  u_n(r, \theta) = (a_n \cos n\theta + b_n \sin n\theta) \left( c_n r^n + \frac{d_n}{r^n} \right) \quad \forall n \in \mathbb{N}^* 
\end{cases}$$

hence any convergent sum of these functions

$u_0$ $u_2$ $u_4$ $u_5$
Some harmonic functions in $\mathbb{R}^3$

- Polynomials

\[
u(x, y, z) = ax^2 + by^2 - (a + b)z^2 + \text{terms in } x, y, z, xy, xz, yz, xyz\]

Extension of 2D harmonic functions

\[
u(x, y, z) = u_{\lambda_1}(x, y)(a_1z + b_1) + u_{\lambda_2}(x, z)(a_2y + b_2) + u_{\lambda_3}(y, z)(a_3x + b_3)\]

Radial harmonic functions

\[
\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \sin \theta \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial u}{\partial \theta}) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2}
\]

\[
u(r, \theta, \phi) = u(r) = ar + b \text{ on } \mathbb{R}^3 \setminus (0, 0, 0)\]
Some harmonic functions in $\mathbb{R}^3$

- **Polynomials**
  \[
  u(x, y, z) = ax^2 + by^2 - (a+b)z^2 + \text{terms in } 1, x, y, z, xy, xz, yz, xyz
  \]

- **Extension of 2D harmonic functions**
Some harmonic functions in $\mathbb{R}^3$

- **Polynomials**

  \[ u(x, y, z) = ax^2 + by^2 - (a+b)z^2 + \text{terms in } 1, x, y, z, xy, xz, yz, xyz \]

- **Extension of 2D harmonic functions**

  \[ u(x, y, z) = u_{\lambda_1}(x, y)(a_1 z + b_1) + u_{\lambda_2}(x, z)(a_2 y + b_2) + u_{\lambda_3}(y, z)(a_3 x + b_3) \]
Some harmonic functions in $\mathbb{R}^3$

- **Polynomials**

  $$u(x, y, z) = ax^2 + by^2 - (a+b)z^2 + \text{terms in } 1, x, y, z, xy, xz, yz, xyz$$

- **Extension of 2D harmonic functions**

  $$u(x, y, z) = u_{\lambda_1}(x, y)(a_1 z + b_1) + u_{\lambda_2}(x, z)(a_2 y + b_2) + u_{\lambda_3}(y, z)(a_3 x + b_3)$$

- **Radial harmonic functions**

  $$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2}$$

  $$u(r, \theta, \varphi) = u(r) =$$
Some harmonic functions in $\mathbb{R}^3$

- **Polynomials**

$$u(x, y, z) = ax^2 + by^2 - (a+b)z^2 + \text{terms in } 1, x, y, z, xy, xz, yz, xyz$$

- **Extension of 2D harmonic functions**

$$u(x, y, z) = u_{\lambda_1}(x, y)(a_1z + b_1) + u_{\lambda_2}(x, z)(a_2y + b_2) + u_{\lambda_3}(y, z)(a_3x + b_3)$$

- **Radial harmonic functions**

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2}$$

$$u(r, \theta, \varphi) = u(r) = \frac{a}{r} + b \quad \text{on } \mathbb{R}^3 \setminus (0, 0, 0)$$
Mean value property and maximum principle

Mean value property:

\[ u(x) = \int_{B(x, r)} u(y) \, dy = \int_{\partial B(x, r)} u(\sigma) \, d\sigma \quad \forall B(x, r) \subset \Omega \]

An harmonic function is the average of its values over every surrounding ball and sphere.

Maximum principle: Let \( u \) an harmonic function. If \( u \in C^2(\Omega) \) and \( u \in C^0(\bar{\Omega}) \), then \( u \) has no extreme values in \( \Omega \).