ODEs and PDEs

- **differential equation**: relationship involving a function \( u \) and (some of) its derivatives.
  - **ordinary diff. eq. (ODE)**: \( u \) depends on only one variable
  - **partial diff. eq. (PDE)**: \( u \) depends on several variables
Partial differential operators

- **Gradient** \( \nabla u(x) = \begin{pmatrix} \frac{\partial u}{\partial x_1}(x) \\ \vdots \\ \frac{\partial u}{\partial x_n}(x) \end{pmatrix} \)

\[ \frac{\partial u}{\partial d}(x) = \nabla u(x) \cdot d \]

- **Divergence** \( \text{div } u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} \)

Also denoted \( \nabla \cdot u \)
Partial differential operators

- **Curl** \( \text{curl } \mathbf{u} = \begin{pmatrix} \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \end{pmatrix} \) also denoted \( \nabla \wedge \mathbf{u} \)

- **Laplacian** \( \Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} \)
Green formulas

Green formulas = integration by parts in $\mathbb{R}^n$

Basic formula:

$$
\int_{\Omega} \frac{\partial u}{\partial x_k} v \, dx = - \int_{\Omega} u \frac{\partial v}{\partial x_k} \, dx + \int_{\partial \Omega} u v (e_k \cdot n) \, ds
$$

where $e_k$ is the unit vector in direction $x_k$.

Derived formulas (among others):

$$
\int_{\Omega} \Delta u \ v \ dx = - \int_{\Omega} \nabla u \cdot \nabla v \ dx + \int_{\partial \Omega} \frac{\partial u}{\partial n} \ v \ ds
$$

$$
\int_{\Omega} u \ \text{div}E \ dx = - \int_{\Omega} \nabla u \ E \ dx + \int_{\partial \Omega} u \ (E \cdot n) \ ds
$$

$$
\text{...}
$$
Some vocabulary

- **order of a PDE**: highest degree of derivation in the PDE
- **linear PDE**: linear w.r.t. $u$ and its derivatives. **Non linear** otherwise

\[
\begin{align*}
&E_1 x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + c(x, y) u = f(x, y) \\
&E_2 \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = f(x, y) \\
&E_3 \frac{\partial u}{\partial t} + \nu \Delta u = 0 \\
&E_4 \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} = f(x, y)
\end{align*}
\]
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- **order of a PDE**: highest degree of derivation in the PDE
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\[(E_1) \quad x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + c(x, y)u = f(x, y)\]

\[(E_2) \quad \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = f(x, y)\]

\[(E_3) \quad \frac{\partial u}{\partial t} + \nu \Delta u = 0\]

\[(E_4) \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} = f(x, y)\]
Some vocabulary

- order of a PDE: highest degree of derivation in the PDE
- linear PDE: linear w.r.t. $u$ and its derivatives. Non linear otherwise

\[(E_1)\hspace{1cm} x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + c(x, y)u = f(x, y) \quad \text{linear, 1st order}\]

\[(E_2)\hspace{1cm} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = f(x, y) \quad \text{nonlinear, 1st order}\]

\[(E_3)\hspace{1cm} \frac{\partial u}{\partial t} + \nu \Delta u = 0 \quad \text{linear, 2nd order}\]

\[(E_4)\hspace{1cm} \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} = f(x, y) \quad \text{nonlinear, 2nd order}\]
Nonlinearity

\[(U \cdot \nabla) U, \quad u(x) u'(x) : \text{the way to turbulence...}\]
Nonlinearity

\[(\mathbf{U} \cdot \nabla) \mathbf{U}, \quad u(x) u'(x)\] : the way to turbulence...
Nonlinearity

\[(U \cdot \nabla)U, \quad u(x) u'(x) : \text{the way to turbulence...}\]

*Sea surface height - 1/60° simulation
DRAKKAR project - Barnier et al., 2015*
Nonlinearity

$(\mathbf{U} \cdot \nabla) \mathbf{U}$, $u(x) u'(x)$: the way to turbulence and chaos...
Nonlinearity

\[(\mathbf{U} \cdot \nabla)\mathbf{U}, \quad u(x) u'(x) : \text{the way to turbulence and chaos...}\]

A system is chaotic if a slight modification in the initial condition implies a large change in the solution.
Nonlinearity

$$(U \cdot \nabla) U \ , \ u(x) \ u'(x)$$: the way to turbulence and chaos...

A system is chaotic if a slight modification in the initial condition implies a large change in the solution

Henri Poincaré (1854-1912)

Edward Lorenz (1917-2008)
Nonlinearity

The historical Lorenz system (1963)

\[
\begin{align*}
\frac{dx}{dt} &= \alpha(y - x) \\
\frac{dy}{dt} &= \beta x - y - xz \\
\frac{dz}{dt} &= -\gamma z + xy
\end{align*}
\]
Nonlinearity

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} &= \rho g_x - \frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \\
- \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} &= \rho g_y - \frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \\
\frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} &= \rho g_z - \frac{\partial p}{\partial z} + \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)
\end{align*}
\]

http://www.chaos-math.org
Some vocabulary

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Some vocabulary

- **order of a PDE**: highest degree of derivation in the PDE
- **linear PDE**: linear w.r.t. $u$ and its derivatives. *Non linear* otherwise

- **time dependent PDE**: involves a time variable. *Steady-state* otherwise

- steady-state PDE + boundary conditions $\rightarrow$ **Boundary value problem**

- time dependent PDE + initial conditions (+ boundary conditions) $\rightarrow$ **Cauchy problem**
Some usual boundary conditions (BCs)

- **Dirichlet** \( u = g \) on \( \partial\Omega \)

- **Neumann** \( \frac{\partial u}{\partial n} = g \) on \( \partial\Omega \)

- **Robin (or Fourier)** \( \frac{\partial u}{\partial n} + ru = g \) on \( \partial\Omega \)

- **Mixed Dirichlet-Neumann**
  \[
  \begin{cases}
    u = g & \text{on } \Gamma_0 \\
    \frac{\partial u}{\partial n} = h & \text{on } \Gamma_1
  \end{cases}
  \]
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\begin{align*}
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  u = g & \text{on } \Gamma_0 \\
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\end{cases}
\end{align*}
\]

Well posedness

A problem (PDE + initial and/or boundary conditions) is **well posed** iff it has a unique solution + continuous dependence w.r.t. its parameters.
Classification of second order linear PDEs

\[ \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u = f \]  

\( (E) \)
Classification of second order linear PDEs

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(\mathbf{x}) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(\mathbf{x}) \frac{\partial u}{\partial x_i} + c(\mathbf{x}) u = f \quad (E)
\]

Associated quadratic form: \( Q_x(X_1, \ldots, X_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(\mathbf{x})X_iX_j \)
Classification of second order linear PDEs

\[ \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u = f \quad (E) \]

Associated quadratic form: \[ Q_x(X_1, \ldots, X_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x)X_iX_j \]

\( (E) \) is:

- **elliptic** at point \( x \) iff \( Q_x \) is definite (positive or negative)
- **parabolic** at point \( x \) iff \( Q_x \) is positive or negative, but not definite
- **hyperbolic** at point \( x \) iff \( Q_x \) is neither definite, nor positive or negative
Classification of second order linear PDEs

\[ \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u = f \quad (E) \]

Associated quadratic form: \( Q_x(X_1, \ldots, X_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x)X_iX_j \)

\( (E) \) is:

- **elliptic** at point \( x \) iff \( Q_x \) is definite (positive or negative)

  Laplace equation: \( \Delta u = 0 \)

- **parabolic** at point \( x \) iff \( Q_x \) is positive or negative, but not definite

  Heat equation: \( \partial_t u - \nu \Delta u = f \)

- **hyperbolic** at point \( x \) iff \( Q_x \) is neither definite, nor positive or negative

  Wave equation: \( \partial_{tt} u - c^2 \Delta u = f \)
Classification of second order linear PDEs

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(\mathbf{x}) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(\mathbf{x}) \frac{\partial u}{\partial x_i} + c(\mathbf{x}) u = f \quad (E)
\]

Associated quadratic form: \(Q_x(X_1, \ldots, X_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(\mathbf{x}) X_i X_j\)

\((E)\) is:

- **elliptic** at point \(\mathbf{x}\) iff \(Q_x\) is definite (positive or negative)

**Laplace equation:** \(\Delta u = 0 \quad \rightarrow \quad Q(X_1, \ldots, X_n) = \sum_{i=1}^{n} X_i^2\)
Classification of second order linear PDEs

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u = f \quad (E)
\]

Associated quadratic form: \( Q_x(X_1, \ldots, X_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x)X_i X_j \)

(E) is:

- **elliptic** at point \( x \) iff \( Q_x \) is definite (positive or negative)
  
  Laplace equation: \( \Delta u = 0 \quad \rightarrow Q(X_1, \ldots, X_n) = \sum_{i=1}^{n} X_i^2 \)

- **parabolic** at point \( x \) iff \( Q_x \) is positive or negative, but not definite

  Heat equation: \( \partial_t u - \nu \Delta u = f \quad \rightarrow Q(X_1, \ldots, X_n, T) = -\nu \sum_{i=1}^{n} X_i^2 \)
Classification of second order linear PDEs

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u = f \quad (E)
\]

AssOCIATED quadratic form: \( Q_x(X_1, \ldots, X_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x)X_iX_j \)

\((E)\) IS:

- **Elliptic** at point \( x \) iff \( Q_x \) is definite (positive or negative)
  
  LAPlace equation: \( \Delta u = 0 \implies Q(X_1, \ldots, X_n) = \sum_{i=1}^{n} X_i^2 \)

- **Parabolic** at point \( x \) iff \( Q_x \) is positive or negative, but not definite
  
  Heat equation: \( \partial_t u - \nu \Delta u = f \implies Q(X_1, \ldots, X_n, T) = -\nu \sum_{i=1}^{n} X_i^2 \)

- **Hyperbolic** at point \( x \) iff \( Q_x \) is neither definite, nor positive or negative
  
  Wave equation: \( \partial_{tt} u - c^2 \Delta u = f \implies Q(X_1, \ldots, X_n, T) = T^2 - c^2 \sum_{i=1}^{n} X_i^2 \)