The goal of these exercises is to introduce some notions related to the time discretization of PDEs: numerical schemes, stability...

**Exercise 1** *Stability of an explicit scheme for a nonlinear ODE*

Let consider the problem:
\[
\begin{aligned}
\left\{ \begin{array}{l}
u'(t) = -u^2(t) + u(t) + 1, \: t > 0 \\
u(0) = 1
\end{array} \right.
\end{aligned}
\]

1.1 Solve this ODE.

1.2 Let an explicit discretization of the ODE using an Euler scheme:
\[
\begin{aligned}
\frac{u_{n+1} - u_n}{\delta t} &= -u_n^2 + u_n + 1, \: t > 0 \\
u_0 &= 1
\end{aligned}
\]

where \(u_n\) is an approximation of \(u(n\delta t)\). By studying the sign of the right-hand side, and of \(\frac{u_{n+1} - l}{u_n - l}\), where \(l = \frac{1 + \sqrt{5}}{2}\), find a stability criterion for this discretization.

**Exercise 2** *Predictor-corrector schemes*

Let consider the ODE/PDE \(\frac{\partial u}{\partial t} = F(u)\). Prove that each of the following time stepping schemes leads to a consistent approximation of the exact solution \(u\), and find its order of accuracy and/or its dominant error term.

2.1 Let consider the following explicit Euler scheme, with an implicit Euler correction:
\[
\begin{aligned}
\tilde{u}_{n+1} &= u_n + \delta t F(u_n) \\
\text{then} \\
u_{n+1} &= u_n + \delta t F(\tilde{u}_{n+1})
\end{aligned}
\]

This scheme can also be written as:
\[
\frac{u_{n+1} - u_n}{\delta t} = F(u_n + \delta t F(u_n))
\]

Let consider \(\varepsilon_n\) the discretization error:
\[
\varepsilon_n = \frac{u(t_{n+1}) - u(t_n)}{\delta t} - F(u(t_n) + \delta t F(u(t_n)))
\]
Prove that \( \varepsilon_n = -\frac{\delta t}{2} u''(t_n) + O(\delta t^2) \), which means that this scheme is first order accurate.

2.2 Let consider now the second order Runge-Kutta scheme:
\[
\begin{align*}
\tilde{u}_{n+1} &= u_n + \frac{\delta t}{2} F(u_n) \\
\text{then} \\
u_{n+1} &= u_n + \delta t F(\tilde{u}_{n+1})
\end{align*}
\]
Prove that it is indeed second order accurate.

2.3 Same question with the explicit Euler scheme with so-called trapezoidal rule (Heun’s method):
\[
\begin{align*}
\tilde{u}_{n+1} &= u_n + \delta t F(u_n) \\
\text{then} \\
u_{n+1} &= u_n + \frac{\delta t}{2} [F(u_n) + F(\tilde{u}_{n+1})]
\end{align*}
\]

Exercise 3 Stability of simple discretization schemes

3.1 Let consider the transport-reaction equation:
\[
\begin{align*}
\frac{\partial u}{\partial t}(x,t) + ru(x,t) + c \frac{\partial u}{\partial x}(x,t) &= 0 \quad x \in \mathbb{R}, \ t > 0 \\
u(x,0) &= u_0(x)
\end{align*}
\]
where \( r \) and \( c \) are given positive constants. Propose a simple explicit discretization scheme for this PDE, and study its stability.

3.2 Same question with the reaction-diffusion equation:
\[
\begin{align*}
\frac{\partial u}{\partial t}(x,t) + ru(x,t) - \nu \frac{\partial^2 u}{\partial x^2}(x,t) &= 0 \quad x \in \mathbb{R}, \ t > 0 \\
u(x,0) &= u_0(x)
\end{align*}
\]
where \( r \) and \( \nu \) are given positive constants.