Exercise 1 - 1-D transport-diffusion equation

Let the PDE:
\begin{align*}
\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} &= 0 & t > 0, x \in \mathbb{R} \\
 u(x,0) &= u_0(x)
\end{align*}

where \( \nu > 0 \) and \( c \) are given coefficients, and \( u_0 \in L^2(\mathbb{R}) \) is assumed to be regular \( (C^1(\mathbb{R}) \) for instance).

1. In a few words, give a qualitative description of the expected behavior of \( u \) (you may also draw an additional sketch). In particular, what is its value at infinity? Do you expect a maximum principle satisfied by \( u \)?

2. Prove that \( E(t) = \frac{1}{2} \int_{\mathbb{R}} u^2(x,t) \, dx \) is a decreasing function of time.

3. Solve the equation. Is it consistent with your answer to the previous question?

   *Hint: note that \( FT^{-1} \left( \hat{u}_0(\xi) \, e^{-2\pi c t \xi} \right)(x) = \int_{\mathbb{R}} \hat{\hat{u}}_0(\xi) \, e^{-2\pi c t \xi} \, e^{2\pi x \xi} \, d\xi = u_0(x - ct) \)

4. Let the discretization (with usual notations):
\[
\frac{u_{j+1}^n - u_j^n}{\delta t} + c \frac{u_{j+1}^n - u_{j-1}^n}{2\delta x} - \nu \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\delta x^2} = 0
\]

   What is its order of accuracy? Study its stability.

   *Hint: prove that the inequality occurring in the calculation reads \([1 - \cos(p\delta x)][A \cos(p\delta x) + B] \leq 0.\)

5. What is the equivalent PDE of the preceding discretization? Is it consistent with the result you found at the preceding question?

   *Hint: transform \( \frac{\partial^2 u}{\partial t^2} \) in terms of \( \frac{\partial^2 u}{\partial x^2}, \frac{\partial^3 u}{\partial x^3} \ldots \)
Exercise 2 - Method of characteristics

Solve the initial value problem:

\[
\begin{aligned}
&\frac{\partial u}{\partial t} - \frac{1}{2} \frac{\partial u}{\partial x} + xu = 0 \quad x \in \mathbb{R}, \ t > 0 \\
u(x, 0) = 2xe^{x^2}
\end{aligned}
\]

Exercise 3 - Lax-Wendroff scheme for the wave equation

Let the 1D wave equation:

\[
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, x \in (a, b), t > 0,
\]

where the celerity \(c\) is given.

Let the auxiliary variables \(v = \frac{\partial u}{\partial x}\) and \(w = \frac{1}{c} \frac{\partial u}{\partial t}\). The wave equation can then be replaced by the system of first order equations:

\[
\begin{aligned}
&\frac{\partial v}{\partial t} = c \frac{\partial w}{\partial x} \\
&\frac{\partial w}{\partial t} = c \frac{\partial v}{\partial x}
\end{aligned}
\]

Similarly to the transport equation, the Lax-Wendroff scheme for this system reads, with standard notations:

\[
\begin{aligned}
v^{n+1}_j &= v^n_j + C \frac{w^{n+1}_j - w^{n-1}_j}{2 \delta x} + \frac{c^2 \delta t^2}{2} \frac{v^{n+1}_{j+1} - 2v^n_j + v^{n-1}_j}{\delta x^2} \\
w^{n+1}_j &= w^n_j + C \frac{v^{n+1}_j - v^{n-1}_j}{2 \delta x} + \frac{c^2 \delta t^2}{2} \frac{w^{n+1}_{j+1} - 2w^n_j + w^{n-1}_j}{\delta x^2}
\end{aligned}
\]

i.e.

\[
\begin{aligned}
v^{n+1}_j &= v^n_j + C \frac{1}{2} (w^{n+1}_{j+1} - w^{n-1}_j) + C \frac{1}{2} (v^{n+1}_{j+1} - 2v^n_j + v^{n-1}_j) \\
w^{n+1}_j &= w^n_j + C \frac{1}{2} (v^{n+1}_{j+1} - v^{n-1}_j) + C \frac{1}{2} (w^{n+1}_{j+1} - 2w^n_j + w^{n-1}_j)
\end{aligned}
\]

where \(C\) is the Courant number.

Let now study the stability of this scheme using the Fourier method.

1. Assuming that \(v^n_j\) and \(w^n_j\) respectively read \(V_0 \exp(ipj \delta x)\) and \(W_0 \exp(ipj \delta x)\), prove that there exists a matrix \(A\), independent of \(j\) and \(n\), such that

\[
\begin{aligned}
v^n_j &= V_n \exp(ipj \delta x) \\
w^n_j &= W_n \exp(ipj \delta x)
\end{aligned}
\]

with

\[
\begin{pmatrix} V_n \\ W_n \end{pmatrix} = A^n \begin{pmatrix} V_0 \\ W_0 \end{pmatrix}
\]

2. A stability condition for the Lax-Wendroff scheme is thus that \(\rho(A) \leq 1\), where \(\rho(A)\) is the spectral radius of \(A\). By computing the characteristic polynomial of \(A\), find this stability criterion.