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An Overview of the Mumford–Shah Problem

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1. Introduction

The *Mumford–Shah functional* is probably one of the best known models in image segmentation. It has been proposed by D. Mumford and J. Shah in their seminal paper [41] and is related to previous discrete models introduced by D. Geman and S. Geman ([35]) and A. Blake and A. Zisserman ([15]). In the Mumford–Shah image reconstruction model we are given an open set Ω in the plane (typically a rectangle) and a function $g : \Omega \to [0, 1]$ representing the grey levels of a picture. Then, one wants to determine a pair (K, u), where $K \subset \overline{\Omega}$ is a compact set representing the contours reconstructed from the discontinuities of g and $u \in C^1(\Omega \setminus K)$ is a smooth approximation of g outside K. The pair (K, u) is obtained by minimizing the functional

$$\mathcal{MS}(K,u) = \int_{\Omega\setminus K} |\nabla u|^2\,dx + \alpha \int_{\Omega\setminus K} |u-g|^2\,dx + \beta\mathcal{H}^1(K\cap\Omega)\,, \qquad (1.1)$$

where α, β are positive constants and $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure. While the term $\int_{\Omega\setminus K} |u-g|^2\,dx$ forces u to be close to the datum g, the integral $\int_{\Omega\setminus K} |\nabla u|^2\,dx$ penalizes strong variations of u, thus ensuring that u is a smooth approximation of g. The interesting feature of the functional (1.1) is that, due to the presence of the term $\mathcal{H}^1(K\cap\Omega)$, whenever g has sharp discontinuities (as it is likely to happen on the edges of the objects in the picture) it is more convenient to insert a contour instead of having a big gradient of u. Finally, α and β are, respectively, scale and contrast parameters.

The *Mumford–Shah functional* (1.1) has an obvious generalization in higher dimension

$$\mathcal{MS}(K,u) = \int_{\Omega\setminus K} |\nabla u|^2\,dx + \alpha\int_{\Omega\setminus K} |u-g|^2\,dx + \beta\mathcal{H}^{N-1}(K\cap\Omega)\,,$$

where Ω is a bounded open set in $\mathbb{R}^N$, $g \in L^\infty(\Omega)$, $\alpha, \beta > 0$, K is a compact subset of $\overline{\Omega}$, $u \in C^1(\Omega \setminus K)$ and $\mathcal{H}^{N-1}$ stands for the $(N-1)$-dimensional Hausdorff measure in $\mathbb{R}^N$. We call *an optimal pair* any minimizer (K,u) of $\mathcal{MS}$. Notice that, given an optimal pair (K,u), any other pair (K',u) which is obtained by adding to or removing from K a set of zero $\mathcal{H}^{N-1}$ measure is also optimal. For this reason it is useful to introduce the concept of *essential* optimal pair. We say that an optimal pair (K,u) is essential if $\mathcal{H}^{N-1}(K \cap B_\varrho(x)) > 0$ for any $x \in K$, $\varrho > 0$. It is not hard to show that, given any optimal pair (K,u), there exists always an essential optimal pair (K',u') such that $\mathcal{H}^{N-1}(K\triangle K') = 0$ and $u' = u$ in $\Omega \setminus (K \cap K')$.

Another variational model used in image segmentation is the *Blake–Zisserman functional*

$$\begin{aligned}\mathcal{BZ}(K_1,K_2,u) = &\int_{\Omega\setminus(K_1\cup K_2)} \left[|\nabla u|^2 + \alpha|u-g|^2\right] dx \\ &+ \beta\mathcal{H}^{N-1}(K_1\cap\Omega) + \gamma\mathcal{H}^{N-1}((K_2\setminus K_1)\cap\Omega)\,,\end{aligned}$$

where K_1 (the jump set) and K_2 (the crease set) are compact subsets of $\overline{\Omega}$, the smooth approximation u of g is in $C(\Omega \setminus K_1) \cap C^2(\Omega \setminus (K_1 \cup K_2))$, $g \in L^\infty(\Omega)$ and α, β, γ are positive constants. The Blake–Zisserman functional allows a more precise segmentation than the Mumford–Shah functional in the sense that also the curvature of the edges of the original picture is approximated. On the other hand, minimizers may not always exist, depending on the values of the parameters β, γ and on the summability assumptions on g (see [21], [22] for existence results and [23] for a counterexample to existence). Here, we shall not discuss the Blake–Zisserman model, but the reader may refer to [24] and [25] for some recent results concerning the regularity of minimizers and to [9] for numerical approximation.

The minimization of the Mumford–Shah and the Blake–Zisserman functional are two examples of a large class of variational problems called by De Giorgi "free discontinuity problems". This terminology refers to the fact that the corresponding functionals are characterized by a competition between volume energies, concentrated on N-dimensional sets, and surface energies, concentrated on $(N-1)$-dimensional sets, whose supports are not fixed a priori. Indeed, as in the case of the two functionals above, the sets where the lower dimensional energies concentrate are the most relevant unknown of the problem.

2. Existence of minimizers for the Mumford–Shah problem

2.1. Preliminary remarks

This section contains an account of the proof of the existence of minimizers for the *Mumford–Shah problem*

$$\text{Min}\left\{\mathcal{MS}(K,u):\ K\subset\overline{\Omega}\text{ compact},\ u\in C^1(\Omega\setminus K)\right\}, \tag{2.1}$$

given by De Giorgi, Carriero and Leaci in [33]. We start by observing that minimizing $\mathcal{M}S$ among all pairs (K,u), with $K\subset\overline{\Omega}$ compact and $u\in C^1(\Omega\setminus K)$ is equivalent to minimizing in the class

$$\mathcal{C}=\left\{(K,u):\ K\subset\overline{\Omega}\text{ compact},\ u\in W^{1,2}(\Omega\setminus K)\right\}. \tag{2.2}$$

In fact, if $(K,u)\in\mathcal{C}$ is a minimizer of $\mathcal{M}S$ in the class $\mathcal{C}$, adding to u a function of the type $\varepsilon\varphi$, with $\varphi\in C^1_0(\Omega\setminus K)$, and letting ε go to zero, we get immediately that u is a weak solution of the equation

$$\int_{\Omega\setminus K}[\langle\nabla u,\nabla\varphi\rangle-\alpha(u-g)\varphi]\,dx=0\qquad\text{for any }\varphi\in C^1_0(\Omega\setminus K)$$

and thus, by standard elliptic regularity results, $u\in L^\infty_{\text{loc}}(\Omega\setminus K)$, hence $u\in W^{2,p}_{\text{loc}}(\Omega\setminus K)$ for any $p<\infty$. Therefore $u\in C^{1,\alpha}(\Omega\setminus K)$ for any $\alpha<1$.

Let us now take a minimizing sequence (K_h,u_h) of $\mathcal{M}S$ in the class $\mathcal{C}$ and let us try to prove the existence of a minimizer by a typical direct argument. Notice that for any $u\in W^{1,2}(\Omega\setminus K)$, setting

$$Tu(x)=\begin{cases}\|g\|_\infty & \text{if }\ u(x)>\|g\|_\infty,\\ u(x) & \text{if }\ -\|g\|_\infty\le u(x)\le\|g\|_\infty,\\ -\|g\|_\infty & \text{if }\ u(x)<-\|g\|_\infty\,,\end{cases} \tag{2.3}$$

then $\mathcal{MS}(K,Tu)\le\mathcal{MS}(K,u)$. Therefore, without loss of generality, we may assume that $\|u_h\|_\infty\le\|g\|_\infty$ for any h. On the other hand, since K_h is a sequence of equibounded compact sets, then, up to a subsequence, we may assume that K_h converges in the Hausdorff metric to a compact $K \subset \overline{\Omega}$, i.e. that

(i) for any $x \in K$ there exists a sequence $x_h \in K_h$ such that $x_h \to x$,
(ii) if $x_h \in K_h$ for any h, then any limit point x of x_h belongs to K.

Moreover, it is easy to prove that the sequence u_h converges weakly in $W^{1,2}_{loc}(\Omega \setminus K)$ to a function $u \in W^{1,2}(\Omega \setminus K)$. At this point, since the two integrals in the definition of $\mathcal{MS}$ are lower semicontinuous, it would be nice to conclude that (K, u) is a minimizer for $\mathcal{MS}$. However, a serious difficulty occurs due to the fact that in general the map $K \mapsto \mathcal{H}^{N-1}(K)$ is not lower semicontinuous with respect to the Hausdorff convergence unless some additional assumptions on the sets K_h are made. To overcome the failure of the direct methods, De Giorgi has proposed a weaker formulation of the Mumford–Shah problem for which the existence of minimizers, based on a lower semicontinuity result of Ambrosio, can be obtained by direct methods. A regularization argument (proved in [33]) then shows that the minimizer of this new functional provides also a minimizer for the original Mumford–Shah problem. But before going in further details, we recall a few facts on functions of bounded variations (shortly BV functions) and rectifiable sets which we shall use in the sequel. A complete exposition on the subject can be found in Chapters 2, 3 and 4 of the book [13] to which we shall constantly refer in the sequel (see also [34]).

2.2. Background

Given an open set Ω in $\mathbb{R}^N$, we denote by $BV(\Omega)$ the space of functions of bounded variation in Ω, i.e. the space of those functions $u \in L^1(\Omega)$ such that the distributional gradient $Du = (D_1, \dots, D_N u)$ is a vector valued Radon measure in Ω, with finite total variation $|Du|(\Omega)$. If $u \in BV(\Omega)$, we say that u is *approximately continuous* at $x \in \Omega$ if there exists $\widetilde{u}(x) \in \mathbb{R}$ such that

$$\lim_{\varrho \to 0} \frac{1}{\varrho^N} \int_{B_\varrho(x)} |u(y) - \widetilde{u}(x)| \, dy = 0 \,,$$

where $B_\varrho(x)$ denotes the open ball with center x and radius ϱ. We denote by C_u the set of all points of approximate continuity of u and by $S_u = \Omega \setminus C_u$ the set of *approximate discontinuity* of u. S_u is a Borel set and $\mathcal{H}^{N-1}$-a.e. $x \in S_u$ is an *approximate jump point*, i.e. there exist a direction $\nu_u(x) \in \mathbb{S}^{N-1}$ and two real numbers $u_+(x), u_-(x)$ such that $u_-(x) < u_+(x)$ and

$$\begin{aligned} &\lim_{\varrho\to 0} \frac{1}{\varrho^N} \int_{B_\varrho^+(x,\nu_u(x))} |u(y) - u_+(x)|\, dy = 0\,, \\ &\lim_{\varrho\to 0} \frac{1}{\varrho^N} \int_{B_\varrho^-(x,\nu_u(x))} |u(y) - u_-(x)|\, dy = 0\,, \end{aligned} \tag{2.4}$$

where $B_\varrho^+(x,\nu_u(x)) = \{y \in B_\varrho(x) : \langle \nu_u(x), y - x\rangle > 0\}$ and $B_\varrho^-(x,\nu_u(x))$ is defined in the obvious way. We denote by J_u the *jump set* of u, i.e. the set of points where (2.4) holds. Since $\mathcal{H}^{N-1}(S_u \setminus J_u) = 0$, in the sequel we shall essentially identify to two sets and refer also to S_u as the jump set. For $\mathcal{L}^N$-a.e. $x \in C_u$ ($\mathcal{L}^N$ stands for the Lebesgue outher measure in $\mathbb{R}^N$) there exists the *approximate gradient* of u at x, which is a vector $\nabla u(x) \in \mathbb{R}^N$ such that

$$\lim_{\varrho\to 0} \frac{1}{\varrho^{N+1}} \int_{B_\varrho(x)} |u(y) - \widetilde{u}(x) - \langle \nabla u, y - x\rangle|\, dy = 0\,.$$

The approximate gradient ∇u turns out to be equal to the derivative of the distributional gradient Du with respect to the Lebesgue measure $\mathcal{L}^N$. Therefore, denoting by $D^a u$ the absolutely continuous part of Du with respect to $\mathcal{L}^N$, we have

$$D^a u = \nabla u \mathcal{L}^N\,.$$

The singular part $D^s u$ of Du can be also split in two mutually singular measures

$$D^j u = D^s u \llcorner S_u, \qquad\qquad D^c u = D^s u \llcorner (\Omega \setminus S_u)\,,$$

where the symbol $\llcorner$ denotes the restriction of a Radon measure to a fixed set. The measure $D^j u$ is called the *jump part* of the derivative Du, while $D^c u$ is called the *Cantor part*. The jump part can be represented as $D^j u = (u_+ - u_-)\nu_u \mathcal{H}^{N-1} \llcorner S_u$, i.e. for any Borel set B in Ω

$$D^j u(B) = \int_{S_u \cap B} (u_+(x) - u_-(x))\nu_u(x)\, d\mathcal{H}^{N-1}\,.$$

On the other hand, the Cantor part does not see the $(N-1)$-dimensional subsets of Ω. More precisely, if $B \subset \Omega$ is a Borel set, σ-finite with respect to $\mathcal{H}^{N-1}$, then $D^c u(B) = 0$. Thus, the distributional gradient of u is decomposed as $Du = \nabla u \mathcal{L}^N + D^c u + (u_+ - u_-)\nu_u \mathcal{H}^{N-1} \llcorner S_u$ and, since the three measures in which Du is split are mutually singular, we have

$$|Du|(\Omega) = \int_\Omega |\nabla u|\, dx + |D^c u|(\Omega) + \int_\Omega |u_+ - u_-|\, d\mathcal{H}^{N-1}\,.$$

Simple examples show that even in one dimension only some of the three part in which Du can be divided explicitely appear. In particular, if $D^j u = D^c u \equiv 0$, then $u \in W^{1,1}(\Omega)$.

The space $BV(\Omega)$ is a Banach space with respect to the norm $\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + |Du|(\Omega)$. Norm convergence, however, is too strong for most applications and indeed we shall mostly use the *weak** convergence. To this aim we recall that a sequence u_h in $BV(\Omega)$ is said to converge weakly* to a function $u \in BV(\Omega)$ if $u_h \to u$ in $L^1(\Omega)$ and the gradients Du_h converge to Du weakly* in the sense of measures, i.e.

$$\lim_{h\to\infty} \int_\Omega \varphi \, dD_i u_h = \int_\Omega \varphi \, dD_i u_h \quad \text{for any } i = 1, \dots, N \text{ and any } \varphi \in C_0(\Omega)\,.$$

In order to provide the right functional setting for the weak formulation of the Mumford–Shah as well as for other free discontinuity problems, De Giorgi and Ambrosio introduced in [31] the space of *special functions of bounded variation* $SBV(\Omega)$. A function $u \in BV(\Omega)$ is said to be in $SBV(\Omega)$ if the Cantor part of Du is zero. Thus, for a special BV function u, we have that $Du = \nabla u \mathcal{L}^N + (u_+ - u_-)\nu_u \mathcal{H}^{N-1} \llcorner S_u$. It can be easily checked that $SBV(\Omega)$ is a closed subspace of the Banach space $BV(\Omega)$. However, since any BV function u can be approximated in the weak* convergence by a sequence of smooth function, SBV is not closed under weak* convergence. The following result, first proved by Ambrosio in [5] (see also [6] and [2] for a different proof), is the main tool to prove the compactness and the lower semicontinuity along minimizing sequences of free discontinuity problems like the Mumford–Shah problem.

Theorem 2.1. *Let Ω be a bounded open set in $\mathbb{R}^N$ and u_h a sequence in $SBV(\Omega) \cap L^\infty(\Omega)$ such that, for some $p > 1$,*

$$\sup_{h\in\mathbb{N}} \Big[\int_\Omega |\nabla u_h|^p \, dx + \|u_h\|_{L^\infty(\Omega)} + \mathcal{H}^{N-1}(S_{u_h}) \Big] < \infty\,.$$

Then, there exists a subsequence u_{h_k} converging weakly in $BV(\Omega)$ to a function $u \in SBV(\Omega)$. Moreover, $\nabla u_{h_k} \to \nabla u$ weakly in $L^p(\Omega)$ and $\mathcal{H}^{N-1}(S_u) \le \liminf_{k\to\infty} \mathcal{H}^{N-1}(S_{u_{h_k}})$.*

We conclude this quick review of the basic material needed for the sequel, by recalling that a set S is said *countably* $(N-1)$*-rectifiable* if there exists a sequence C_h of compact subsets of $(N-1)$-dimensional manifolds M_h of class C^1 such that $S = \cup_h C_h \cup N_0$, where N_0 is a set of zero $\mathcal{H}^{N-1}$-measure. A suitable notion of *approximate tangent plane* can be given so that an approximate tangent plane π_x exists at $\mathcal{H}^{N-1}$-a.e. point x of a rectifiable set. However, the precise definition of π_x is not needed in the sequel. Here we limit ourselves to recall that, for $\mathcal{H}^{N-1}$-a.e. point x in each of the compact sets C_h in which S can be decomposed, the approximate tangent plane π_x coincides with the classical tangent plane at x to the underlying manifold M_h containing C_h. Moreover, for any $x \in S$ such that π_x exists, we have

$$\lim_{\varrho\to 0}\frac{1}{\varrho^{N+1}}\int_{S\cap B_\varrho(x)} \operatorname{dist}^2(y,\pi_x)\, d\mathcal{H}^{N-1}(y) = 0\,. \tag{2.5}$$

Notice also that if $u \in BV(\Omega)$, the jump set S_u is countably $(N-1)$-rectifiable and for $\mathcal{H}^{N-1}$-a.e. point $x \in S_u$ the direction $\nu_u(x)$ along which u jumps is orthogonal to the approximate tangent plane to S_u at x.

The following lemma is a special case of a classical result concerning k-dimensional densities of Radon measures.

Lemma 2.2. *Let μ be a Radon measure in $\mathbb{R}^N$, $t > 0$ and $B \subset \mathbb{R}^N$ a Borel set such that*

$$\limsup_{\varrho\to 0}\frac{\mu(B_\varrho(x))}{\varrho^{N-1}} \ge t \quad \textit{for any } x \in B\,.$$

Then $\mu(B) \ge t\omega_{N-1}\mathcal{H}^{N-1}(B)$, where ω_{N-1} is the $\mathcal{L}^{N-1}$ measure of the unit ball in $\mathbb{R}^{N-1}$.

The following result is a simple consequence of the previous lemma.

Corollary 2.3. *Let f be a locally summable function from $\mathbb{R}^N$. Set*

$$\Lambda = \left\{ x \in \mathbb{R}^N : \limsup_{\varrho\to 0}\frac{1}{\varrho^{N-1}}\int_{B_\varrho(x)} |f(y)|\, dy > 0 \right\}.$$

Then, $\mathcal{H}^{N-1}(\Lambda) = 0$.

Proof. Let us define, for any Borel set $E \subset \mathbb{R}^N$, $\mu(E) = \displaystyle\int_E |f(y)|dy$. Then, we can set $\Lambda = \cup_j \Lambda_j$, where, for any $j \in \mathbb{N}$,

$$\Lambda_j = \left\{ x \in B_j(0) : \limsup_{\varrho\to 0}\frac{\mu(B_\varrho(x))}{\varrho^{N-1}} > \frac{1}{j} \right\}.$$

Since $\mu(\Lambda_j) < \infty$ for any j, from Lemma 2.2 we get that also $\mathcal{H}^{N-1}(\Lambda_j) < \infty$, hence $\mathcal{L}^N(\Lambda_j) = 0$ and thus $\mu(\Lambda_j) = 0$. Therefore, Lemma 2.2 again implies that $\mathcal{H}^{N-1}(\Lambda_j) = 0$, hence $\mathcal{H}^{N-1}(\Lambda) = 0$. □

2.3. Existence of minimizers

Let us come back to the Mumford–Shah functional. In order to prove the existence of minimizers of problem (2.1), De Giorgi suggested to minimize the following *relaxed Mumford–Shah functional*

$$MS(u) = \int_\Omega |\nabla u|^2\,dx + \alpha\int_\Omega |u-g|^2\,dx + \beta\mathcal{H}^{N-1}(S_u) \tag{2.6}$$

in the class of $SBV(\Omega)$ functions.

His idea was to deal with a simpler object, just depending on the function u, and then to recover the set of contours K by taking the set of discontinuity $\overline{S}_u$. The key point in this approach is that the right functional space where to set the problem is $SBV(\Omega)$ and not the entire space $BV(\Omega)$, which would be too large. Indeed, given any function $g \in L^2(\Omega)$, one can construct a sequence u_h in $BV(\Omega)$, converging to g in L^2, such that for any h the derivative of each u_h is made up only by the Cantor part. Therefore the infimum of $MS(u)$ on $BV(\Omega)$ is trivially zero. On the other hand, since we are mostly interested in finding a set of contours K, it is natural to retain only those BV functions whose discontinuity set may provide in some way the set K. Moreover, the compactness and lower semicontinuity Theorem 2.1 of Ambrosio immediately gives the existence of minimizers for the relaxed functional $MS(u)$.

Proposition 2.4. *Let Ω be a bounded open set of $\mathbb{R}^n$, g a function from $L^\infty(\Omega)$, $\alpha, \beta > 0$. Then, the problem*

$$\operatorname{Min}\{MS(u) : u \in SBV(\Omega)\}\,, \tag{2.7}$$

where MS is the functional defined in (2.6)*, has always a solution.*

Proof. Let $u_h \in SBV(\Omega)$ be a minimizing sequence for MS. For any h we set $v_h = Tu_h$, where T denotes the truncation operator defined in (2.3). Then, it is easy to check that $v_h \in SBV(\Omega)$, $\nabla v_h = \nabla u_h \chi_{\{|u_h|\le\|g\|_\infty\}}$ and that $S_{v_h} \subset S_{u_h}$ for any h. Moreover $MS(v_h) \le MS(u_h)$, and thus v_h is a minimizing sequence satisfying the assumptions of Theorem 2.1. Therefore, with no loss of generality, we may assume that v_h converge weakly* in BV to a function $u \in SBV(\Omega)$ and that the conclusions of Theorem 2.1 hold. In particular, since $v_h \to u$ in $L^1(\Omega)$ and is bounded in L^∞, v_h converges to u also in $L^2(\Omega)$ and, since ∇v_h converges to ∇u weakly in $L^2(\Omega)$, then $\displaystyle\int_\Omega |\nabla u|^2 dx \le \liminf_{h\to\infty}\int_\Omega |\nabla v_h|^2 dx$. Therefore we get that $MS(u) \le \liminf_{h\to\infty} MS(v_h)$, hence u is a minimizer. □

The next step is now to prove that the two minimum problems, (2.7) and (2.1), are equivalent and that any minimizer of $MS(u)$ provides an optimal pair $(\overline{S}_u, u)$ for the original problem (2.1). This is by no means obvious, once we keep in mind that the discontinuity set S_u of a SBV function can be in general very wild and that in particular S_u can be even dense in Ω. However, a relationship between the two minimum problems can be easily deduced from the following general result concerning SBV functions.

Proposition 2.5. *Let Ω be a bounded open in $\mathbb{R}^N$ and let $K \subset \mathbb{R}^N$ be a closed set such that $\mathcal{H}^{N-1}(K \cap \Omega) < \infty$. If $u : \Omega \to \mathbb{R}$ is a function from $W^{1,1}(\Omega \setminus K) \cap L^\infty(\Omega \setminus K)$, then u belongs to $SBV(\Omega)$ and $\mathcal{H}^{N-1}(S_u \setminus K) = 0$.*

As an immediate consequence of Proposition 2.5 we have

$$\min_{u \in SBV(\Omega)} MS(u) \leq \inf_{(K,u) \in \mathcal{C}} \mathcal{MS}(K, u)\,, \tag{2.8}$$

where $\mathcal{C}$ is defined in (2.2).

To prove the opposite inequality, namely that for any minimizer u of MS the pair $(\overline{S}_u, u)$ is an optimal pair for $\mathcal{MS}$, we have to show that certain pathological behaviours of BV functions cannot occur when dealing with minimizers of MS. In other words, we must prove some sort of regularity for the discontinuity set of a minimizer of problem (2.7). This approach, which has been introduced in the paper [33], leads to a proof of existence through regularity, a strategy used in many classical variational problems in order to prove existence. The regularity property needed in our case, known in the literature as *Ahlfors regularity*, is contained in the following statement.

Theorem 2.6. *There exist three constants $\vartheta, \gamma, \varrho_0$, depending only on N, α, β and $\|g\|_\infty$, such that, if $u \in SBV(\Omega)$ is a minimizer for MS, $x \in S_u$, $B_\varrho(x) \subset \Omega$ and $\varrho < \varrho_0$, then*

$$\vartheta \varrho^{N-1} \leq \mathcal{H}^{N-1}(S_u \cap B_\varrho(x)) \leq \gamma \varrho^{N-1}\,. \tag{2.9}$$

The meaning of the estimate (2.9) is clear. When u is a minimizer, each time we fix a point in S_u, the amount of jump set around this point is never too much nor too little and so S_u is not too diffuse nor too sparse in Ω. Let us now show how the existence of solutions for the original Mumford–Shah problem can be easily derived from this estimate and then we give a hint of how the proof of (2.9) goes.

Existence of minimizers for the Mumford–Shah problem. Actually, only the inequality on the left in (2.9) is needed. First, notice that, by continuity, (2.9) holds also for any $x \in \overline{S}_u \cap \Omega$. Then, let us define, for any Borel set $E \subset \mathbb{R}^N$, $\mu(E) = \mathcal{H}^{N-1}(E \cap S_u)$ and set $B = (\overline{S}_u \setminus S_u) \cap \Omega$. We have

$$\limsup_{\varrho \to 0} \frac{\mu(B_\varrho(x))}{\varrho^{N-1}} \geq \vartheta \quad \text{for every } x \in B\,.$$

Therefore, Lemma 2.2 implies that

$$\mu\left((\overline{S}_u \setminus S_u) \cap \Omega\right) \geq \omega_{N-1} \vartheta \mathcal{H}^{N-1}\left((\overline{S}_u \setminus S_u) \cap \Omega\right)$$

and thus we get $\mathcal{H}^{N-1}\left((\overline{S}_u \setminus S_u) \cap \Omega\right) = 0$. From this equality, we have immediately that $(\overline{S}_u, u)$ is an optimal pair. Infact, since $|D^s u|(\Omega \setminus \overline{S}_u) = 0$, we have that $u \in W^{1,1}(\Omega \setminus \overline{S}_u)$, hence $u \in W^{1,2}(\Omega \setminus \overline{S}_u)$. Moreover, $MS(u) = \mathcal{MS}(\overline{S}_u, u)$, hence the result follows from (2.8). □

Notice that, since the estimate from below in (2.9) holds for any $x \in \overline{S}_u \cap \Omega$, the optimal pair $(\overline{S}_u, u)$ obtained with the above proof is also essential.

Let us now give a sketch of the proof of the estimate (2.9). The inequality on the right follows by a simple comparison argument. In fact, let u be a minimizer for MS. Then, $\|u\|_\infty \leq \|g\|_\infty$, otherwise we would immediately get that $MS(Tu) < MS(u)$, where T is defined in (2.3).

Let us fix a ball $B_\varrho(x)$ and let us compare the functional MS at the two functions u and $w = u(1 - \chi_{B_\varrho(x)})$. Since $MS(u) \leq MS(w)$, we get

$$\begin{aligned}
&\int_\Omega |\nabla u|^2 dy + \alpha \int_\Omega |u-g|^2 dy + \beta \mathcal{H}^{N-1}(S_u) \\
&\qquad \leq \int_{\Omega \setminus B_\varrho(x)} |\nabla u|^2 dy + \alpha \int_{B_\varrho(x)} |g|^2 dy + \alpha \int_{\Omega \setminus B_\varrho(x)} |u-g|^2 dy \\
&\qquad\quad + \beta \mathcal{H}^{N-1}\left(S_u \cap (\Omega \setminus B_\varrho(x))\right) + N\omega_N \beta \varrho^{N-1}
\end{aligned}$$

and thus

$$\begin{aligned}
\int_{B_\varrho(x)} |\nabla u|^2 dy + \alpha \int_{B_\varrho(x)} |u-g|^2 dy + \beta \mathcal{H}^{N-1}(S_u \cap B_\varrho(x)) \\
\leq \alpha \int_{B_\varrho(x)} |g|^2 dy + N\omega_N \beta \varrho^{N-1} \\
\leq \alpha \|g\|_\infty^2 \varrho^N + N\omega_N \beta \varrho^{N-1}\,.
\end{aligned}$$

Therefore, from the last inequality, we have that if $\varrho < 1$

$$\mathcal{H}^{N-1}(S_u \cap B_\varrho(x)) \le \frac{\alpha\omega_N}{\beta}\|g\|_\infty^2 \varrho^N + N\omega_N \varrho^{N-1} \le \gamma\varrho^{N-1}\,,$$

hence the right hand side inequality in (2.9) is proved.

The proof of the estimate from below in (2.9) is much more delicate and we cannot reproduce it here. We limit ourselves to recall that this estimate is based on the following decay lemma, stated in a slightly different form in [33].

Lemma 2.7 (Decay)**.** *There exists a constant C_1, depending only on N, α, β and $\|g\|_\infty$, such that, for any minimizer u of MS and any $\tau \in (0,1)$, there exist $\varepsilon, \eta > 0$ with the property that, if $B_\varrho(x) \subset \Omega$, with $\varrho < \eta$, and*

$$\mathcal{H}^{N-1}(S_u \cap B_\varrho(x)) < \varepsilon\varrho^{N-1}\,,$$

then

$$\int_{B_{\tau\varrho}(x)} |\nabla u|^2\,dy + \mathcal{H}^{N-1}(S_u \cap B_{\tau\varrho}(x)) \le C_1\tau^N \left[\int_{B_\varrho(x)} |\nabla u|^2\,dy + \mathcal{H}^{N-1}(S_u \cap B_\varrho(x))\right].$$

The above lemma is proved by a typical *blow-up* argument, showing that if in the ball $B_\varrho(x)$ the amount of jump of u, $\mathcal{H}^{N-1}(S_u \cap B_\varrho(x))$, is very small compared to ϱ^{N-1} then u is very close to a harmonic function, hence $\int_{B_r(x)} |\nabla u|^2 dy$ decays like r^N for a smaller radius r. In this case, one can prove that also $\mathcal{H}^{N-1}(S_u \cap B_r(x))$ decays like r^N.

Iterating the above estimate in smaller and smaller balls, it is not difficult to prove that if the amount of S_u inside the ball $B_\varrho(x)$ is below a certain critical value $\varepsilon_0\varrho^{N-1}$, then actually $S_u \cap B_{\varrho/2}(x) = \emptyset$ and from this fact the proof of the estimate from below in (2.9) immediately follows.

Another interesting consequence of the lower bound in (2.9) is the fact that if K is a compact set contained in $\overline{S}_u \cap \Omega$, then

$$\mathcal{H}^{N-1}(K) = \lim_{\varepsilon \to 0} \frac{\mathcal{L}^N\left(I_\varepsilon(K)\right)}{2\varepsilon}\,, \tag{2.10}$$

where $I_\varepsilon(K) = \{x \in \mathbb{R}^N : \operatorname{dist}(x,K) < \varepsilon\}$. The limit on the right hand side of (2.10), whenever exists, is known as the $(N-1)$-dimensional *Minkowski content* of the set K. It is not hard to see that for any countably $(N-1)$-rectifiable set S, its Minkowski content is always greater than or equal to

$\mathcal{H}^{N-1}(S)$ and the fact that here we have the equality can be viewed as a sort of (mild) regularity of the jump set S_u.

Different proofs of the existence of minimizers for the Mumford–Shah problem are available in the literature. For instance, Morel and Solimini proved (see [40]) the existence result in dimension 2 by a direct method argument. They use the fact that if K_h is a sequence of equibounded compact sets converging in the Hausdorff metric to K and if the number of connected components of each K_h is also equibounded, then, by Golab's theorem, $\mathcal{H}^1(K) \leq \liminf_{h\to\infty} \mathcal{H}^1(K_h)$. Their argument goes as follows. First, for any $n \in \mathbb{N}$ they prove the existence of a minimizing pair (K_n, u_n) for the Mumford–Shah functional under the constraint that the number of connected components of K is less than or equal to n. Then, they show that any such minimizer (K_n, u_n) satisfies an estimate of the type (2.9). Thus, by means of this extra information, they are able to show that, up to a subsequence, (K_n, u_n) converges to a pair (K, u) in the sense that $K_n \to K$ in the Hausdorff metric and $\mathcal{H}^1(K) \leq \liminf_{n\to\infty} \mathcal{H}^1(K_n)$, u_n converges to u weakly in $W^{1,2}_{\mathrm{loc}}(\Omega \setminus K)$. From these two facts they immediately get that (K, u) is a minimizing pair.

Another 2-dimensional proof of the existence of minimizers has been given by Dal Maso, Morel and Solimini in [28]. They show that the lower semicontinuity of the map $K \mapsto \mathcal{H}^1(K)$ holds along sequences satisfying a suitable *uniform concentration property*. However, the proof of the existence of such minimizing sequences is rather complicate.

The same idea of choosing good minimizing sequences along which the lower semicontinuity of the $\mathcal{H}^{N-1}$ measure holds also inspires a recent proof of Maddalena and Solimini (see [37]), which works in any dimension. Finally, let us mention that by no means the minimizers of the Mumford–Shah functional are unique. Simple examples show that uniqueness fails even in one dimension.

3. Regularity of global and local minimizers

Once we have proved the existence of minimizers u for the functional MS, i.e. that the pair $(\overline{S}_u, u)$ is an essential optimal pair for the original Mumford–Shah functional $\mathcal{MS}$, the next step is to investigate which regularity we may expect for u and for the jump set $\overline{S}_u$. At this regard, Mumford and Shah stated in [41] a precise conjecture, which they were able to prove only in very special situations and which is strongly supported by numerical evidence.

Conjecture (Mumford–Shah)**.** *Let* (K, u) *be an optimal essential pair for* $\mathcal{MS}$*. Then* $K \cap \Omega$ *is locally the union of finitely many* $C^{1,1}$ *arcs. Moreover, the set* $K \cap \Omega$ *may have only two kinds of singularities: either a "crack tip" or a "triple junction".*

Here, by a *crack tip*, we mean a point $x \in K \cap \Omega$ such that x is the endpoint of a $C^{1,1}$ arc, while $x \in K \cap \Omega$ is a *triple junction* if three $C^{1,1}$ arcs meet at x forming equal angles of $2\pi/3$. The simplest example of a set K with a triple junction singularity is given by a *"propeller"*, i.e. the union of three half lines meeting at one point and forming equal angles.

The Mumford–Shah conjecture is far from being solved. However partial results in this direction have been obtained by Ambrosio and Pallara in [7], Ambrosio, Fusco and Pallara in [11], [12], by Bonnet in [16], [17], [18], by David in [26], David and Semmes in [27] and by Ambrosio, Fusco and Hutchinson in [10].

Before describing these results let us make some preliminary remarks.

Let u be a function from $SBV(\Omega)$ and let $B_\varrho(x_0)$ be a ball contained in Ω. If we set

$$u_\varrho(y) = \varrho^{-1/2} u(x_0 + \varrho y)\,,$$

then $u_\varrho \in SBV(\Omega_\varrho)$, where $\Omega_\varrho = \varrho^{-1}(\Omega - x_0)$. Moreover, for any $\sigma \in (0, 1)$,

$$\mathcal{H}^{N-1}(S_{u_\varrho} \cap B\sigma) = \varrho^{1-N} \mathcal{H}^{N-1}\left(S_u \cap B_{\sigma\varrho}(x_0)\right),$$

$$\int_{B_\sigma} |\nabla u_\varrho|^2 dy = \varrho^{1-N} \int_{B_{\sigma\varrho}(x_0)} |\nabla u|^2 dx\,.$$

Thus, both the Dirichlet integral and the area term in the functional MS rescale by the same factor ϱ^{N-1}. On the other hand, since when u is a minimizer we have $\|u\|_\infty \le \|g\|_\infty$, the extra term $\int_{B_\varrho(x_0)} |u - g|^2 dx$ decays like ϱ^N and therefore, from the point of view of regularity, is negligible with respect to the Dirichlet integral and the area term. For this reason, we shall drop this term in the sequel and, after multiplying u by a suitable constant, we shall also assume that $\beta = 1$. Thus, we will refer our discussion to the (simpler) functional

$$F(u, A) = \int_A |\nabla u|^2 \, dx + \mathcal{H}^{N-1}(S_u \cap A)\,,$$

where A is an open subset of Ω and u is a function from $SBV_{\rm loc}(\Omega)$. However we warn the reader that all the regularity results that we are going to present for the functional F can be extended, with some technical complications, also to the Mumford–Shah functional MS.

We recall the notion of local minimizer of the functional F.

Let $\Omega \subset \mathbb{R}^n$ be an open set; we say that a function $u \in SBV_{\rm loc}(\Omega)$ is a *local minimizer* in Ω if $F(u,A) < \infty$ for any open set $A \subset\subset \Omega$ and

$$F(u,A) \le F(v,A) \tag{3.1}$$

for all $v \in SBV_{\rm loc}(\Omega)$, such that $\{u \neq v\} \subset\subset A$. When u satisfies (3.1) for any open subset $A \subset \mathbb{R}^N$ we say that u is a *global minimizer*.

With this definition in mind, the Mumford–Shah conjecture can be rephrased in terms of local minimizers of the functional F, but passing from MS to F does not really simplify the problem of regularity.

The following theorem gives some special examples of local minimizers. The minimality of the functions considered in the theorem can be proved by using the theory of calibrations for the Mumford–Shah functional recently developed by Alberti, Bouchitté and Dal Maso (see [1]).

Theorem 3.1. *Let Ω be an open subset of $\mathbb{R}^N$ and $u : \Omega \to \mathbb{R}$.*

(i) *If u is a harmonic function such that*

$$\left(\sup_\Omega u - \inf_\Omega u\right) \|\nabla u\|_{L^\infty(\Omega)} \le 1\,, \tag{3.2}$$

then u is a local minimizer in Ω.

(ii) *Let $\Omega = U \times I$, where U is an open subset of $\mathbb{R}^{N-1}$ and $I \subset \mathbb{R}$ is an open interval, and let $u = a$ in $\Omega \cap \{x_N > 0\}$, $u = b$ in $\Omega \cap \{x_N < 0\}$. If $\mathcal{L}^1(I) < (b-a)^2$, then u is a local minimizer in Ω.*

(iii) *Let u be a function jumping along a propeller T and taking constant values a, b, c in the three connected components of $\mathbb{R}^2 \setminus T$. Then, u is a minimizer in any ball $B_R(x)$ such that $R \le \frac{1}{2} \min\{|a-b|^2, |b-c|^2, |c-a|^2\}$.*

Proof of (i). We shall give here a proof of (i) due to Chambolle and which does not make use of calibrations.

Let us assume without loss of generality that $\inf_\Omega u = 0$ and that $\sup_\Omega u = M$. Then, fix an open set $A \subset\subset \Omega$ and consider the following minimum problem

$$\operatorname{Min}\left\{F(v,A) :\, v \in SBV_{\rm loc}(\Omega),\, v = u \ \text{ in } \ \Omega \setminus \overline{A}\right\}\,.$$

With the same argument used to prove Proposition 2.4 one can easily get that there exists a minimizer v for the above problem and that $0 \le v \le M$. Setting $v_\varepsilon = v + \varepsilon(u - v)$, for $\varepsilon \neq 0$, and noticing that $S_{v_\varepsilon} \cap A = S_v \cap A$, from the minimality of v we have that $F(v, A) \le F(v_\varepsilon, A)$ and thus

$$\int_A |\nabla v|^2 \, dx \le \int_A |\nabla v + \varepsilon(\nabla u - \nabla v)|^2 \, dx \,.$$

From this inequality we easily get that

$$\int_A \langle \nabla v, \nabla u - \nabla v \rangle \, dx = 0 \,. \tag{3.3}$$

To prove the assertion, since A is arbitrary, it is enough to show that $F(u, A) \le F(v, A)$, i.e.

$$\int_A |\nabla u|^2 \, dx \le \int_A |\nabla v|^2 \, dx + \mathcal{H}^{N-1}(S_v \cap A) \,. \tag{3.4}$$

Using (3.3), it is clear that (3.4) is equivalent to

$$\int_A \langle \nabla u, \nabla u - \nabla v \rangle \, dx \le \mathcal{H}^{N-1}(S_v \cap A) \,. \tag{3.5}$$

To prove this inequality we observe that $Dv = \nabla v \mathcal{L}^N + (v_+ - v_-)\nu_v \mathcal{H}^{N-1} \llcorner S_v$ and u is harmonic; therefore, using the Gauss–Green formulas, which still hold in BV, we get

$$\begin{aligned}
&\int_A \langle \nabla u, \nabla u - \nabla v \rangle \, dx \\
&\qquad = \sum_{i=1}^N \int_A \frac{\partial u}{\partial x_i} \, dD_i(u - v) - \int_{S_v \cap A} (v_+ - v_-) \langle \nabla u, \nu_v \rangle \, d\mathcal{H}^{N-1} \\
&\qquad = -\int_A \Delta u (u - v) \, dx - \int_{S_v \cap A} (v_+ - v_-) \langle \nabla u, \nu_v \rangle \, d\mathcal{H}^{N-1} \\
&\qquad \le \mathcal{H}^{N-1}(S_v \cap A) M \|\nabla u\|_{L^\infty(A)} \,.
\end{aligned}$$

From this inequality and from the assumption (3.2), (3.5) immediately follows. $\square$

Concerning global minimizers of F, we have the following

Conjecture (De Giorgi)**.** *The only nonconstant global minimizer in $\mathbb{R}^2$ of F is the function u, given in polar coordinates by*

$$u(\varrho, \vartheta) = \sqrt{\frac{2\varrho}{\pi}} \sin(\vartheta/2) \quad \varrho \ge 0, \ -\pi < \vartheta < \pi \,. \tag{3.6}$$

It is clear that in the above conjecture uniqueness must be understood up to rigid motions, sign change and addition of constants. Recently, Bonnet and David in [19], gave a rather long proof of the fact that the function u in (3.6) is a local minimizer. Uniqueness is still an open problem.

A very interesting approach to the problem of characterizing global minimizers was taken by Bonnet, who introduced in two dimensions a slightly different minimality condition. Let us recall his definition.

We say that a pair (K, u), where $K \subset \mathbb{R}^2$ is a closed subset and $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2 \setminus K)$, is a *global Bonnet minimizer* if

$$\int_{B_{2R}(x_0)\setminus K} |\nabla u|^2\, dx + \mathcal{H}^1(K \cap B_{2R}(x_0)) \leq \int_{B_{2R}(x_0)\setminus C} |\nabla v|^2\, dx + \mathcal{H}^1(C \cap B_{2R}(x_0))$$

whenever (C, v) is another pair with C closed in $\mathbb{R}^2$, $v \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2 \setminus C)$, such that $K \setminus B_R(x_0) = C \setminus B_R(x_0)$, $u = v$ outside $B_R(x_0)$ and for any two points in $\mathbb{R}^2 \setminus (K \cup B_R(x_0))$ lying in two different components of $\mathbb{R}^2 \setminus K$ they lie also in two different components of $\mathbb{R}^2 \setminus C$.

Notice that the notion of global minimizer given by Bonnet is weaker than the one that we have introduced before. However, he was able to classify all such global minimizers (K, u) satisfying the additional assumption that K is connected.

Theorem 3.2. *Let (K, u) be a global minimizer such that K is connected. Then K can be either the empty set, or a line, or a propeller or a half-line. In the first three cases u is constant in the connected components of $\mathbb{R}^2\setminus K$, while in the last case u coincides, up to a constant, with the function defined in* (3.6)*, where the polar coordinates system is centered at the tip of the half-line K.*

Let us now state the regularity results proved in [7], [11] and [12]. Notice that, in the two dimensional case essentially the same result has been proved by David with a completely different proof (see [26]). While David's proof uses mostly ideas from harmonic and complex analysis, the proof of the regularity in [7], [11] (which works in any dimension) is very much inspired to the proof of the regularity for minimal surfaces and area-minimizing varifolds.

Theorem 3.3. *Let Ω be an open set in $\mathbb{R}^N$ and let $u \in SBV_{\mathrm{loc}}(\Omega)$ be a local minimizer of F in Ω (or a minimizer of MS in Ω). Then, there exists a set $\Sigma \subset \overline{S}_u \cap \Omega$, relatively closed in Ω, with $\mathcal{H}^{N-1}(\Sigma) = 0$, such that $\overline{S}_u \cap \Omega \setminus \Sigma$ is a hypersurface of class $C^{1,\alpha}$ for any $\alpha < 1$ (of class $C^{1,1}$, if $N = 2$).*

Clearly, the gap between this result and the Mumford–Shah conjecture is still big. Infact, in two dimensions, Theorem 3.3 gives no information about the singularities of the set $\overline{S}_u$, and says only that the 1-dimensional Haudorff measure of Σ is zero. Actually, David's regularity result in this respect is slightly better, since he is able to prove that the Hausdorff dimension of Σ is strictly less than one. Yet, we are still far from proving that Σ is a zero dimensional set, with a locally finite number of points. On the other hand the fact that $\overline{S}_u \cap \Omega$ is $C^{1,1}$ outside Σ is in agreement with what has been conjectured by Mumford and Shah.

The same remark applies in higher dimension. In this case, another conjecture set by De Giorgi in [30] says that the singular set Σ should have locally finite $\mathcal{H}^{N-2}$ measure. Again, there is a gap of one dimension between this conjecture and what is stated in the regularity theorem above.

As we have noticed before, the proof of Theorem 3.3 has many analogies with De Giorgi's regularity proof of minimal surfaces (see, for instance, [29]) and with Allard's regularity proof of area-minimizing varifolds (see [3]). Indeed, the situation considered in the above theorem is even closer to the one considered in Brakke's book ([20]), where varifolds whose mean curvature is only in L^1 are studied.

Let us now give a sketchy description of how the proof (which is rather long) goes.

Just to have a first look at the problem we may derive the Euler–Lagrange equation. To this aim, let us fix a vector field $\eta \in C^1_0(\Omega;\mathbb{R}^N)$ and $\varepsilon \neq 0$ sufficiently small, so that the map $\Phi(y) = y + \varepsilon\eta(y)$ is a diffeomorphism from Ω into itself. By comparing the functional F at the local minimum u and at the function u_ε, where $u_\varepsilon(x) = u(\Phi^{-1}(x))$, after some more or less standard calculations we get

$$\int_{\Omega\setminus\overline{S}_u} \big[|\nabla u|^2 \mathrm{div}\eta - 2\langle \nabla u, \nabla u \cdot \nabla\eta\rangle\big]\,dx + \int_{S_u} \mathrm{div}^{S_u}\eta\, d\mathcal{H}^{N-1} = 0\,, \quad (3.7)$$

where $\mathrm{div}^{S_u}\eta$ is the tangential divergence of η on S_u, which can be defined as in the case of smooth manifolds, since S_u is a countably $(N-1)$-rectifiable set and thus for $\mathcal{H}^{N-1}$-a.e. $x \in S_u$ there exists the approximate tangent plane. Roughly speaking, equation (3.7) says that for a local minimizer u

the Dirichlet integral controls the mean curvature of the jump set S_u. In particular, from (3.7) we get that u is harmonic in $\Omega \setminus \overline{S}_u$, namely that

$$\begin{cases} \Delta u & \text{in } \ \Omega \setminus \overline{S}_u \\ \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \ \overline{S}_u \cap \Omega \end{cases} \tag{3.8}$$

and that, if $A \subset\subset \Omega$ is an open set such that $\overline{S}_u \cap A$ is the graph of a smooth function ϕ, i.e. that up to a rotation $\overline{S}_u \cap A = \{x = (z, \phi(z)) : z \in D\}$, with D open and ϕ smooth, then

$$\operatorname{div}\left(\frac{\nabla \phi}{\sqrt{1 + |\nabla \phi|^2}}\right) = \big[|\nabla u|^2\big]^{\pm} \quad \text{in } D\,, \tag{3.9}$$

where $\big[|\nabla u|^2\big]^{\pm}$ denotes the jump of $|\nabla u|^2$ across $\overline{S}_u \cap A$ (notice that if $\overline{S}_u \cap A$ is $C^{1,\alpha}$ for some $\alpha > 0$, then by (3.8) ∇u has a continuous extension on both sides of $\overline{S}_u \cap A$).

Equations (3.8) and (3.9) can be used only when we know already that $\overline{S}_u \cap A$ is $C^{1,\alpha}$ and in this case they easily imply further regularity both on u and $\overline{S}_u \cap A$. In fact, by a bootstrap argument one can prove (see [12]) that u has a C^∞ extension on each side of $\overline{S}_u \cap A$ and $\overline{S}_u \cap A$ is a C^∞ hypersurface. It is interesting to recall that De Giorgi conjectured that whenever $\overline{S}_u \cap A$ is of class $C^{1,\alpha}$ then it is also analytic. This conjecture is still open, although a significant step in this direction has been obtained in a recent paper ([36]) by Leoni and Morini.

Thus, the crucial point for regularity is to prove the (partial) $C^{1,\alpha}$ regularity of $\overline{S}_u \cap \Omega$. To explain how this result is obtained in [7] and [11] we must introduce the two relevant quantities for the problem, the *rescaled Dirichlet integral*

$$\mathcal{D}(x, \varrho) = \frac{1}{\varrho^{N-1}} \int_{B_\varrho(x)} |\nabla u|^2 \, dy$$

and the *rescaled flatness* of $\overline{S}_u$ in the ball $B_\varrho(x)$

$$\mathcal{A}(x, \varrho) = \min_{\pi \in \mathcal{P}} \frac{1}{\varrho^{N+1}} \int_{\overline{S}_u \cap B_\varrho(x)} \operatorname{dist}^2(y, \pi) \, d\mathcal{H}^{N-1}(y)\,,$$

where $\mathcal{P}$ is the set of all $(N-1)$-dimensional hyperplanes π in $\mathbb{R}^N$. In fact, the $C^{1,\alpha}$ regularity of the jump set of a local minimizer of F can be described in terms of the decay of the two above quantities.

Theorem 3.4. *Let u be a local minimizer of the functional F in the open set A, and let $c_0 > 0$, $s \in [0, 1/2]$. If*

$$\mathcal{D}(x,\varrho) + \mathcal{A}(x,\varrho) \le c_0 \varrho^s$$

for any ball $B_\varrho(x) \subset A$ and any $x \in \overline{S}_u \cap A$, then $\overline{S}_u \cap A$ is a $C^{1,s/2}$ embedded hypersurface.

At this point, a typical decay estimate asserts that if the sum $\mathcal{D}(x,\varrho) + \mathcal{A}(x,\varrho)$ is sufficiently small in a certain ball $B_\varrho(x)$, with $x \in \overline{S}_u \cap \Omega$, then in smaller balls of radius r it decays like $r^{1/2}$.

Theorem 3.5. *There exist two positive numbers R_0, ε_0 depending only on N, such that if u is a local minimizer of F in Ω, $x \in \overline{S}_u \cap \Omega$, $B_\varrho(x) \subset \Omega$, $\varrho < R_0$ and*

$$S(x,\varrho) = \mathcal{D}(x,\varrho) + \mathcal{A}(x,\varrho) < \varepsilon_0 \,,$$

then, for all $r \in (0,\varrho)$,

$$S(x,r) \le C r^{1/2} \left[1 + S(x,\varrho)\right] \,,$$

where the constant C depends only on N.

The above theorem readily leads to the proof of the regularity theorem.

Proof of Theorem 3.3. Let us set

$$R = \left\{ x \in \overline{S}_u \cap \Omega : \, S(x,\varrho) < \varepsilon_0 \text{ for some } \varrho < \min\{R_o, \operatorname{dist}(x,\partial\Omega)\} \right\} .$$

The set R is relatively open in $\overline{S}_u \cap \Omega$ and, by Theorems 3.4 and 3.5, $\overline{S}_u \cap R$ is a $C^{1,1/4}$ embedded surface. Let $A \subset \Omega$ be an open set such that $\overline{S}_u \cap A \subset \overline{S}_u \cap R$. Then, from equation (3.8), we get that ∇u has a $C^{0,1/4}$ extension on both sides of $\overline{S}_u \cap A$. At this point we use equation (3.9). In fact, since $\overline{S}_u \cap A$, locally, is the graph of a $C^{1,1/4}$ function ϕ, satisfying (3.9) with a continuous right hand side, more or less standard elliptic regularity result will imply that ϕ is $C^{1,\alpha}$ for any $\alpha < 1$ (or that ϕ is $C^{1,1}$, if $N = 2$). Therefore, to conclude the proof we need only to show that the set $\Sigma = \overline{S}_u \cap \Omega \setminus R$ is $\mathcal{H}^{N-1}$ negligible. To this aim let us observe that if $x \in \Sigma$, then at least one of the two following relations holds

$$\limsup_{\varrho \to 0^+} \frac{1}{\varrho^{N-1}} \int_{B_\varrho(x)} |\nabla u|^2 \, dy \ge \varepsilon_0 \,, \tag{3.10}$$

or

$$\limsup_{\varrho \to 0^+} \frac{1}{\varrho^{N+1}} \min_{\pi \in \mathcal{P}} \int_{\overline{S}_u \cap B_\varrho(x)} \operatorname{dist}^2(y,\pi) \, d\mathcal{H}^{N-1}(y) \ge \varepsilon_0 \,. \tag{3.11}$$

By Corollary 2.3, the set of points where (3.10) holds has zero $\mathcal{H}^{N-1}$ measure. Moreover, also the set of points where (3.11) holds is $\mathcal{H}^{N-1}$ negligible. In fact if $x \in \overline{S}_u \cap \Omega$ and if the approximate tangent plane at x, π_x exists, then by (2.5) we have

$$\limsup_{\varrho \to 0^+} \frac{1}{\varrho^{N+1}} \min_{\pi \in \mathcal{P}} \int_{\overline{S}_u \cap B_\varrho(x)} \mathrm{dist}^2(y,\pi) d\mathcal{H}^{N-1}_y \leq \lim_{\varrho \to 0^+} \frac{1}{\varrho^{N+1}} \int_{\overline{S}_u \cap B_\varrho(x)} \mathrm{dist}^2(y,\pi_x) d\mathcal{H}^{N-1}_y = 0\,.$$

Therefore, the set of points where (3.11) holds is contained in the set of points where the tangent plane does not exists, which has zero $\mathcal{H}^{N-1}$ measure, since $\overline{S}_u$ is rectifiable. $\square$

Notice that the above proof shows that in fact

$$\Sigma = \left\{ x \in \overline{S}_u \cap \Omega : \limsup_{\varrho \to 0^+} \mathcal{D}(x,\varrho) > 0 \quad \text{or} \quad \limsup_{\varrho \to 0^+} \mathcal{A}(x,\varrho) > 0 \right\}.$$

It is interesting to compare this characterization of the singular set Σ of $\overline{S}_u \cap \Omega$ with the Mumford–Shah conjecture in two dimensions. In fact, at a crack tip point x we have

$$\lim_{\varrho \to 0^+} \mathcal{A}(x,\varrho) = 0, \qquad \text{but} \qquad \limsup_{\varrho \to 0^+} \mathcal{D}(x,\varrho) > 0\,, \tag{3.12}$$

and at a triple junction point x we have the opposite situation, namely

$$\lim_{\varrho \to 0^+} \mathcal{D}(x,\varrho) = 0, \qquad \text{but} \qquad \limsup_{\varrho \to 0^+} \mathcal{A}(x,\varrho) > 0\,. \tag{3.13}$$

Therefore, it would be nice, as a first step in the direction of proving the Mumford–Shah conjecture, to show that, at least when $N = 2$, only the situations considered in (3.12) and (3.13) may occur and that there are no singular points in $\overline{S}_u \cap \Omega$ where both $\mathcal{D}(x,\varrho)$ and $\mathcal{A}(x,\varrho)$ do not vanish as ϱ goes to zero.

A small step in the direction of understanding the singular points of $\overline{S}_u \cap \Omega$ is taken in the paper [10] where it is proved (in any dimension) that, setting

$$\Sigma' = \left\{ x \in \Sigma : \lim_{\varrho \to 0^+} \mathcal{D}(x,\varrho) = 0 \right\},$$

then the Hausdorff dimension of Σ' is less than or equal to $N - 2$. In the same paper it is also shown that if $x \in \Sigma'$, then there exists a sequence

$\varrho_h \to 0$ such that

$$\mathcal{H}^{N-1} \llcorner \frac{S_u - x}{\varrho_h} \to \mathcal{H}^{N-1} \llcorner C \quad \text{weakly* in the sense of measures,}$$

where C is an Almgren area minimizing cone (see [10, Section 4] and [4]). A characterization of these cones proved in [42], implies that in two dimensions C is a propeller.

4. Final remarks

Many papers have been recently devoted to the numerical approximation of the Mumford–Shah functional. Here, we just describe an approach which is probably the most successful from the point of view of applications. To this aim, let us recall the notion of Γ-convergence, introduced by De Giorgi and Franzoni in [32].

Let (X, d) be a metric space and let $F_h, F : X \to [0, +\infty]$ be functions. We say that F_h Γ-converge to F if the following two conditions are satisfied:

(i) for any sequence x_h in X converging to x, then $\liminf_{h\to\infty} F_h(x_h) \geq F(x)$;

(ii) for any $x \in X$ there exists a sequence x_h converging to x such that $\limsup_{h\to\infty} F_h(x_h) \leq F(x)$.

Notice that the Γ-limit F of F_h is uniquely determined by (i) and (ii). The importance of this notion relies on the fact that it implies the convergence of minimizers of the approximating functionals to minimizers of the limiting functional. More precisely, if F_h Γ-converges to F in X and if there exists a compact set $K \subset X$ such that

$$\min_{x\in K} F_h(x) = \min_{x\in X} F_h(x)\,, \tag{4.1}$$

then $\min_{x\in X} F_h(x)$ converges to $\min_{x\in X} F(x)$. Moreover, if x_h is a sequence of minimizers of F_h, up to a subsequence, x_h converges to a minimizer x of F.

Going back to the Mumford–Shah functional MS, it is clear that dealing numerically with the term $\mathcal{H}^{N-1}(S_u)$ in (2.6) can be rather complicate. Thus, one would like to approximate the functional MS with a sequence of (simpler) functionals, not involving surface energies.

A similar problem is solved by a theorem due to Modica and Mortola (see [39] and [38]) concerning the approximation of the perimeter $P(E;\Omega)$ of a set E in a given open set Ω. To explain their result, let us denote by W a "double well" potential, i.e. a real function with two absolute minima, convex at $\pm\infty$; a simple example of such a function is given by $W(t) = t^2(1-t)^2$. Let us fix $\varepsilon > 0$ and set

$$MM_\varepsilon(v) = \int_\Omega \Big(\varepsilon|\nabla v|^2 + \frac{W(v)}{\varepsilon}\Big)dx \quad v \in W^{1,2}(\Omega)\,.$$

Then, in [39] it is proved that the functionals MM_ε Γ-converge with respect to the $L^2(\Omega)$ convergence to a functional F, where $F(v)$ is equal to $P(E;\Omega)/3$ if $v = \chi_E$ and is equal to $+\infty$ otherwise.

In the same spirit of the above result, De Giorgi suggested to introduce the functional

$$AT_\varepsilon(v,u) = \int_\Omega v^2\left[|\nabla u|^2 + \alpha(u-g)^2\right]dx + \frac{\beta}{2}\int_\Omega\Big(\varepsilon|\nabla v|^2 + \frac{W(v)}{\varepsilon}\Big)dx\,,$$

where $v \in W^{1,2}(\Omega)$, $0 \le v \le 1$, $uv \in W^{1,2}(\Omega)$, and $AT_\varepsilon(v,u) = +\infty$ otherwise. Then, Ambrosio and Tortorelli proved in [8] that AT_ε converges in $L^2(\Omega)\times L^2(\Omega)$ to the functional

$$F(v,u) = \int_\Omega \left[|\nabla u|^2 + \alpha(u-g)^2\right]dx + \beta\mathcal{H}^{N-1}(S_u),\ \text{if } u \in SBV(\Omega),\ v \equiv 1\,,$$

and $F(v,u) = +\infty$ otherwise.

To understand why this remarkable approximation result holds one should observe that if $(v_\varepsilon, u_\varepsilon)$ is a sequence of minimizers of AT_ε, then v_ε must be very close to 1 when ε goes to zero, since $W(t)$ is positive except for $t = 1$, where $W(t) = 1$. On the other hand, near a point where ∇u_ε is very big v_ε must be close to zero. Thus the functions v_ε must be close to 1 for a large portion of Ω and must rapidly decrease to zero near the discontinuity points of u, where u is the limit of u_ε. Therefore, while the functions u_ε approximate a minimizer u of the functional MS, the level sets $\{v_\varepsilon = 0\}$ approximate its jump set S_u. It is also clear that it is much easier to deal numerically with the functionals AT_ε than with the original Mumford–Shah functional MS and this is the reason why most numerical approximations of MS actually reduce to approximating AT_ε for small values of ε. Indeed, these numerical procedures can be justified rigorously by using a result of Bellettini and Coscia ([14]) who proved that the functional MS can be also approximated by a sequence of discretized functionals $AT_{\varepsilon,h(\varepsilon)}$, where the mesh size $h(\varepsilon)$ goes to zero faster than ε.

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