# Active contours: from tracking to capturing through level-set 

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## 1 Introduction

An image can be considered as an application of a domain of $\mathbb{R}^{2}$ (or $\mathbb{R}^{3}$ in the case of three-dimensional images) with values in a subset of $\mathbb{R}$ (or $\mathbb{R}^{3}$ for color images). Many algorithms of edge detection, shape recognition, denoising, are based partial differential models.

In this short lecture we consider the Eulerian representation of region of interest in images. We begin by giving the main principles of the Eulerian description of interfaces and what we can derive from it as geometrical information. Then we develop few algorithms of active contours that were developed up to last years. We will explicit numerical tools involved in the implementation of these algorithms.

## 2 Segmentation methods using level-set in image analysis

### 2.1 Introduction

To represent an interface defining two moving regions of interest, several representations are possible. In particular, we may consider a parametrization of the interface (for simplicity assume that this is a curve $\Gamma_{t}$ ). We call it a Lagrangian setting: let $\gamma:[0, M] \times[0, T] \rightarrow \gamma(r, t)$ a parametrization of $\Gamma_{t}$ advected by the continuous medium velocity $\gamma(r, t)=X\left(t ; \gamma_{0}(r)\right)$ or equivalently the solution of differential system:

$$
\begin{cases}\partial_{t} \gamma(r, t)=u(\gamma(r, t), t), & r \in[0, M], t \in] 0, T]  \tag{1}\\ \gamma(r, 0)=\gamma_{0}(r), & r \in[0, M]\end{cases}
$$

In practice, to capture a region, we will therefore consider a discretization of $[0, M]$ given by $\left(r_{i}\right)_{i=0 \ldots N}$ and then move $\gamma\left(r_{i}, t\right)$ to enclose the target region. The fact that the spacing between the points may vary as a result of the curve deformation could cause over or under-sampling. Moreover, with this representation, handling topology changes of the contour can be cumbersome.

### 2.2 Level set framework

We now change our representation of the interface to circumvent the pitfalls of the Lagrangian representation. We represent $\Gamma_{t}$, that we from now on consider as closed ${ }^{1}$ by introducing an auxiliary function $\phi: \Omega \times[0, T] \rightarrow \mathbb{R}$ such that

$$
\Gamma_{t}=\{x \in \Omega, \quad \phi(x, t)=0\} .
$$

The bounded domain delimited by $\Gamma_{t}$, that we refer as the region interior, is $\Omega_{t}^{-}=\{x \in \Omega, \quad \phi(x, t)<$ $0\}$ and its exterior as $\Omega_{t}^{+}=\{x \in \Omega, \quad \phi(x, t)>0\}$. As $\phi(\gamma(r, t), t)=0$ on $[0, M] \times[0, T]$, and $\partial_{t} \gamma=u(\gamma, t)$, we get differentiating with respect to $t$

$$
\partial_{t} \phi(\gamma(r, t), t)+u(\gamma(r, t), t) \cdot \nabla \phi(\gamma(r, t), t)=0
$$

The Level Set method [?, ?] corresponds to consider a initial function $\phi_{0}$ representing $\Gamma_{0}$ and to look for a function $\phi$ which verifies the transport equation on the whole domain:

$$
\begin{cases}\partial_{t} \phi+u \cdot \nabla \phi=0 & \text { on } \Omega \times] 0, T[  \tag{2}\\ \phi=\phi_{0} & \text { on } \Omega \times\{0\} .\end{cases}
$$

Usually $\phi_{0}$ is chosen as a signed distance to interface:

$$
\phi_{0}(x)= \begin{cases}-\operatorname{dist}\left(x, \Gamma_{0}\right) & \text { if } x \text { belongs to the interior of } \Gamma_{0} \\ \operatorname{dist}\left(x, \Gamma_{0}\right) & \text { if } x \text { belongs to the exterior of } \Gamma_{0}\end{cases}
$$

[^0]Remark 1 1. It is remarkable that a smooth function of time and space can be a very singular evolution curve: melting of curves, or splitting of a curve. Is that we have added a dimension to the problem, and in this new space, these situations are not unique! The price to pay is to work with this additional dimension: instead of a $1 D$ discretization, we will consider a discretization of the whole space. Nevertheless, we will see that there are methods to restrict the calculations to an area near the contour (narrow-band) .
2. The level-set method gives a direct way of knowing if a point in space is inside or outside of the contour, by inspecting the sign of $\phi$ at this point. This is generally not trivial with the other representation. On the contrary, access the coordinates of points of the contour is not at all natural, and we'll try to get around this need as possible.

With the choice of sign of $\phi_{0}$ above fact, that we will now assume systematically, the outer normal to the area enclosed by $\Gamma_{t}$ and its curvature are expressed at each of its points by

$$
n(x)=\frac{\nabla \phi}{|\nabla \phi|} \quad \kappa(x)=\operatorname{div} \frac{\nabla \phi}{|\nabla \phi|}
$$

The general proof can be found in the book [?], pages 354 to 357 . We justify the formula of curvature in two dimensions. Consider a regular parametrization $s \rightarrow \gamma(s)$ of $\Gamma_{t}$ such that, by following the curve along the direction of $\partial_{s} \gamma$, we have $(\phi<0)$ on our left. The curvature is expressed in terms of the parameterization as

$$
\kappa=\frac{\left[\partial_{s} \gamma, \partial_{s s} \gamma\right]}{\left|\partial_{s} \gamma\right|^{3}}
$$

By differentiating the identity $\phi(\gamma(s))=0$ we have $\partial_{s} \gamma \cdot \nabla \phi=0$. Differentiating $\partial_{s} \gamma \cdot \frac{\nabla \phi}{|\nabla \phi|}=0$ we have

$$
\partial_{s s} \gamma \cdot \frac{\nabla \phi}{|\nabla \phi|}+\partial_{s} \gamma \cdot\left(D\left[\frac{\nabla \phi}{|\nabla \phi|}\right] \partial_{s} \gamma\right)=0
$$

As $\partial_{s} \gamma \perp \frac{\nabla \phi}{|\nabla \phi|}$ we have due to the chosen curve orientation, $\left[\partial_{s} \gamma, \frac{\nabla \phi}{|\nabla \phi|}\right]<0$, therefore

$$
\left[\partial_{s} \gamma, \partial_{s s} \gamma\right]=\left|\partial_{s} \gamma\right| \partial_{s} \gamma \cdot\left(D\left[\frac{\nabla \phi}{|\nabla \phi|}\right] \partial_{s} \gamma\right)
$$

In dimension two, if we denote as $\nabla \times \phi$ the curl of phi $\phi$, obtained by a $+\frac{\pi}{2}$ rotation to the vector $\nabla \phi$, the corresponding tangent vector is

$$
\tau(x)=\frac{\nabla \times \phi}{|\nabla \phi|}
$$

Therefore

$$
\kappa=\frac{\nabla \times \phi}{|\nabla \phi|}\left(D\left[\frac{\nabla \phi}{|\nabla \phi|}\right] \frac{\nabla \times \phi}{|\nabla \phi|}\right)
$$

The prof is achieved by the following lemma that the reader could prove ( $\mathbb{S}^{1}$ is the unit circle).
Lemma 1 Let $u, v: \Omega \subset \mathbb{R}^{2} \rightarrow \mathbb{S}^{1}$, of class $\mathcal{C}^{1}$, such that $u \cdot v=0$ on $\Omega$. Then $[D u] v \cdot v=\operatorname{div} u$.

### 2.3 Set operations expressed with level set functions

Let two bounded open sets $\Omega_{1}$ and $\Omega_{2}$. If we already have two level set functions to capture those sets, say $\phi_{1}$ and $\phi_{2}$, such that $\phi_{i}<0$ in $\Omega_{i}$, then $\min \left(\phi_{1}, \phi_{2}\right)$ is a level set function for $\Omega_{1} \cup \Omega_{2}$ and $\max \left(\phi_{1}, \phi_{2}\right)$ for $\Omega_{1} \cap \Omega_{2}$. As $\phi_{i}$ is a level set function for $\Omega_{i},-\phi_{i}$ plays the same role for $\Omega_{i}^{c}$. Therefore, $\max \left(-\phi_{1}, \phi_{2}\right)$ is a level set function for $\Omega_{2} \backslash \Omega_{1}$. At last $\min \left(\max \left(-\phi_{1}, \phi_{2}\right), \max \left(\phi_{1},-\phi_{2}\right)\right)$ is a level set function for the symmetric difference $\Omega_{1} \Delta \Omega_{2}$.

### 2.4 General principles of image segmentation and active contours

Consider an image (grayscale to start), that is a function defined on the domain $\Omega, \mathbb{I}: \Omega \rightarrow \mathbb{R}$. To identify areas in this image, a first idea is to look at the histogram of the gray levels of the latter, and segment the image into functions thresholds on the histogram. We will therefore consider as belonging to the same region of interest pixels having an intensity within a certain band. The problem with this type of segmentation is that it does not produce in general a connected region, and is very sensitive to noise. We therefore seek methods of segmentation sufficiently robust to noise and low contrast images. Among the existing segmentation methods, except the threshold described above, one can list the active contours, deformable models, growth models ... We are interested in this part in the active contour models, of which we present the general principles.

It amounts to move a curve (or a surface for a 3D image) so that it fits the contours of the object that we want to identify in the image. In order to move $\Gamma_{t}$, in Lagrangian representation we usually seek to minimize a functional of the following kind:

$$
\begin{equation*}
F(\gamma(t))=\alpha \int_{0}^{M}\left|\partial_{r} \gamma(r, t)\right| d r+\beta \int_{0}^{M}\left|\partial_{r r}^{2} \gamma(r, t)\right|^{2} d r-\lambda \int_{0}^{M}|\nabla \mathbb{I}(\gamma(r, t))|^{2} d r, \tag{3}
\end{equation*}
$$

where $\alpha \geq 0, \beta \geq 0, \lambda>0$ are parameters. The first two measure the smoothness of the curve, the last will move the $\gamma$ curve to areas of strong gradients of the image.

To write the dynamics of movement, rather than writing a derivative with respect to the curve, using our velocity field we try to move the curve $\gamma$ so as to minimize $F$ as $t$ increases. On has

$$
\frac{d}{d t} F(\gamma(t))=d F_{\gamma(t)}\left(\partial_{t} \gamma(t)\right)=d F_{\gamma(t)}(u(\gamma(t)))=\nabla F(\gamma(t)) \cdot u(\gamma(t))
$$

We see that choosing $u(\gamma)=-\rho \nabla F(\gamma(t))$, where $\rho>0$ is a descent parameter, will produce decreasing $F$. This is the "snakes" method introduced by Kass, Witkin, Terzopoulos in 1988. The general principle of these methods is to identify regions that are defined by areas of high variations the gradient of the image. Furthermore, if there are multiple objects to be captured, if the starting area encircles these objects, then the contour will always gather the union of these objects and therefore, as is, this method can not capture multiple objects. This is a serious caveat of the method. More generally we consider a decreasing function $g:[0,+\infty) \rightarrow \mathbb{R}^{+}$such that $g(r) \rightarrow 0$ for $r \rightarrow \infty$, for instance $g(s)=\frac{1}{1+s^{2}}$. We consider here the case $\beta=0$ and therefore the energy boils to:

$$
\begin{equation*}
F(\gamma(t))=\alpha \int_{0}^{M}\left|\partial_{r} \gamma(r, t)\right| d r+\lambda \int_{0}^{M} g(|\nabla \mathbb{I}(\gamma(r, t))|) d r, \tag{4}
\end{equation*}
$$

A remark raised after a few years in the development of the method is the dependence on parameterization of the curve. Noticed by Caselles, Kimmel and Sapiro in 1997, that is explained by the fact that if we define a new parameterization of the curve by a change of variable $r=\phi(s)$ with $\phi:[0, L] \rightarrow[0, M]$, $\phi^{\prime}>0$ we get:

$$
\int_{0}^{M}\left|\partial_{r} \gamma(r, t)\right| d r=\int_{0}^{L}\left|\partial_{s} \gamma(\phi(s), t)\right| d s
$$

which does not depend on parametring, whereas

$$
\int_{0}^{M} g(|\nabla \mathbb{I}(\gamma(r, t))|) d r=\int_{0}^{L} g(|\nabla \mathbb{I}(\gamma(\phi(s), t))|) \phi^{\prime}(s) d s
$$

Which is completely dependent on the parametrization (nb: if you take a square in the first term of energy as in the articles cited, this term also depends on the setting). The trick used by Caselles, Kimmel and Sapiro is simply to write an energy considering the metric of the curve itself, so that it is invariant under change of parametrization:

$$
\int_{0}^{M} g(|\nabla \mathbb{I}(\gamma(r, t))|)\left|\partial_{r} \gamma(r, t)\right| d r=\int_{0}^{L} g(|\nabla \mathbb{I}(\gamma(\phi(s), t))|)\left|\partial_{r} \gamma(\phi(s), t)\right| d s
$$



FIGURE 1: Snakes (Kass, Witkin, and Terzopoulos 1988)

The general form of the energy is now:

$$
\begin{equation*}
F(\gamma(t))=\int_{0}^{M} g(|\nabla \mathbb{I}(\gamma(r, t))|)\left|\partial_{r} \gamma(r, t)\right| d r, \tag{5}
\end{equation*}
$$

since indeed, energy (4) may be expressed under this form by replacing $g$ by $\alpha+\lambda g$. Let us compute the flow generated by such an energy. After some elementary calculations, we find:

$$
\begin{equation*}
\frac{d}{d t} F(\gamma(t))=\int_{0}^{M}[g(\nabla \mathbb{I}(\gamma)) \kappa n+(\nabla(g(\nabla \mathbb{I}))(\gamma) \cdot n) n] \cdot u(\gamma) d r \tag{6}
\end{equation*}
$$

where $n$ is the outward normal, which gives as evolution of $\gamma$ :

$$
\begin{equation*}
\partial_{t} \gamma=-\rho[g(\nabla \mathbb{I}(\gamma)) \kappa n+(\nabla(g(\nabla \mathbb{I}))(\gamma) \cdot n) n] . \tag{7}
\end{equation*}
$$

The application of a discretized version of this algorithm to follow lips motion, in the original article of 1988, is depicted on figure 1. Problems could occur on sharp edges, see figure 2. Algorithm is sensitive to noise, and cannot detect more than one object, see figure 3 .

We now translate this expression to Eulerian coordinates. First is a curve is evolving with normal velocity $V$, that is:

$$
\partial_{t} \gamma=V n
$$

then the transport quartions ont the level set function reads:

$$
\partial_{t} \phi+V n \cdot \nabla \phi=0
$$

which as $n$ is a unit vector field collinear to $\nabla \phi$, gives

$$
\partial_{t} \phi=-V|\nabla \phi|
$$

Thus the equation on $\phi$ becomes (for $\rho=1$ ):

$$
\partial_{t} \phi=g(\nabla \mathbb{I})|\nabla \phi| \operatorname{div} \frac{\nabla \phi}{|\nabla \phi|}+\nabla(g(\nabla \mathbb{I})) \cdot \nabla \phi
$$



FIGURE 2: Snakes, case with sharp edges


FIGURE 3: Snakes, case with two objects and noise

### 2.5 Active contours and level set: one step beyond

### 2.5.1 Computation of volume and surface integrals

The implicit representation can be used to compute approximate surface and volume integrals.
Lemma 2 Let $\phi: \mathbb{R}^{d} \rightarrow \mathbb{R}$ be Lipschitz on $\mathbb{R}^{d}$ and $g: \mathbb{R}^{d} \rightarrow \mathbb{R}$ integrable. Assume there exists $\eta_{0}>0$ such that $\operatorname{ess} \inf _{|\phi|<\eta_{0}}|\nabla \phi|>0$. Then for $\left.\eta \in\right] 0, \eta_{0}[$,

$$
\int_{|\phi(x)|<\eta} g(x) d x=\int_{-\eta}^{\eta} \int_{\phi(x)=\nu} g(x)|\nabla \phi|^{-1} d \sigma d \nu
$$

Proof. In [?], proposition 3 page 118, it is shown under such assumptions that

$$
\frac{d}{d s}\left(\int_{\phi>s} g(x) d x\right)=-\int_{\phi=s} g|\nabla \phi|^{-1} d \sigma \quad \text { a.e. } s
$$

The above result easily follows by setting $s=-t$ and taking $\phi$ and $-\phi$ in this formula, and then summing up the two obtained identities after integrating them between 0 and $\eta$.

A less rigorous but more intuitive proof amounts to write the volume element in a neighborhood of a point $x$ as $d x=d \sigma \times d h$, where $d h$ is along the normal $\frac{\nabla \phi}{|\nabla \phi|}$ to the level-set of $\phi$ through $x$, and by remarking that

$$
\nu \pm d \nu:=\phi\left(x \pm d h \frac{\nabla \phi}{|\nabla \phi|}\right)=\phi(x) \pm d h|\nabla \phi|+O\left(d h^{2}\right)
$$

from which we get $d x=|\nabla \phi|^{-1} d \sigma d \nu$.
From now on we assume the following regularity for $\phi$ :

$$
\left(H_{\phi}\right) \quad \forall t \in[0, T], \forall f \in \mathcal{C}_{c}\left(\mathbb{R}^{n}\right), \quad s \rightarrow \int_{\{|\phi(x, t)|<s\}} f(x) d x \text { est } \mathcal{C}^{1} \text { au voisinage de } s=0
$$

Let $\mathcal{M}\left(\mathbb{R}^{d}\right)$ be the space of bounded measures on $\mathbb{R}^{d}$, that is, of linear continuous forms on the space of bounded continuous functions. Then we have the following:

Proposition 1 Let $r \rightarrow \zeta(r)$ be a continuous cut-off function, that is with support in $[-1,1]$, and such that $r \rightarrow \frac{1}{\varepsilon} \zeta\left(\frac{r}{\varepsilon}\right)$ converges to $\delta_{0}$ in $\mathcal{M}(\mathbb{R})$. Then under assumption $\left(H_{\phi}\right)$, when $\varepsilon \rightarrow 0$,

$$
\frac{1}{\varepsilon} \zeta\left(\frac{\phi}{\varepsilon}\right)|\nabla \phi| \rightharpoonup \delta_{\{\phi=0\}} \quad \text { in } \mathcal{M}\left(\mathbb{R}^{d}\right)
$$

Proof. For a continuous $g(r)$, we have by assumption,

$$
\lim _{\varepsilon \rightarrow 0} \int_{\mathbb{R}} \frac{1}{\varepsilon} \zeta\left(\frac{r}{\varepsilon}\right) g(r) d r=g(0) .
$$

Applying this, for $f \in \mathcal{C}_{c}\left(\mathbb{R}^{d}\right)$, to

$$
g(r)=\int_{\{\phi=r\}} f d \sigma .
$$

We have

$$
\lim _{\varepsilon \rightarrow 0} \int_{\mathbb{R}} \frac{1}{\varepsilon} \zeta\left(\frac{r}{\varepsilon}\right) \int_{\{\phi=r\}} f d \sigma d r=\int_{\{\phi=0\}} f d \sigma,
$$

so that

$$
\lim _{\varepsilon \rightarrow 0} \int_{\mathbb{R}} \int_{\{\phi=r\}} \frac{1}{\varepsilon} \zeta\left(\frac{\phi}{\varepsilon}\right) f d \sigma d r=\int_{\{\phi=0\}} f d \sigma .
$$

As $\frac{1}{\varepsilon} \zeta\left(\frac{\phi}{\varepsilon}\right)$ is vanishing outside $|\phi|<\varepsilon$, using Lemma 2, we get

$$
\begin{aligned}
& \int_{\mathbb{R}} \int_{\{\phi=r\}} \frac{1}{\varepsilon} \zeta\left(\frac{\phi}{\varepsilon}\right) f d \sigma d r=\int_{-\varepsilon}^{\varepsilon} \int_{\{\phi=r\}} \frac{1}{\varepsilon} \zeta\left(\frac{\phi}{\varepsilon}\right) f d \sigma d r=\int_{|\phi(x)|<\varepsilon} \frac{1}{\varepsilon} \zeta\left(\frac{\phi}{\varepsilon}\right) f|\nabla \phi| d x \\
&=\int_{\mathbb{R}^{d}} \frac{1}{\varepsilon} \zeta\left(\frac{\phi}{\varepsilon}\right) f|\nabla \phi| d x
\end{aligned}
$$

which means finally, for all $f \in \mathcal{C}_{c}\left(\mathbb{R}^{d}\right)$,

$$
\lim _{\varepsilon \rightarrow 0} \int_{\mathbb{R}^{d}} \frac{1}{\varepsilon} \zeta\left(\frac{\phi}{\varepsilon}\right)|\nabla \phi| f(x) d x=\int_{\{\phi=0\}} f(x) d \sigma
$$

We therefore hereby justified that $\delta_{\{\phi=0\}}$ can be approximated by $|\nabla \phi| \frac{1}{\varepsilon} \zeta\left(\frac{\phi}{\varepsilon}\right)$ for small $\varepsilon$.
Remark 2 This approximation has however to be handled with care. Indeed it amounts to replace a purely geometrical object, namely the measure supported on a curve, $\delta_{\{\phi=0\}}$ (which depends only on this curve but not on the chosen $\phi$ to capture it), by a function that does depend on the scale of $\phi$ (for instance $\{2 \phi=0\}$ or $\{\phi=0\}$ are the same curve, whereas $|2 \nabla \phi| \frac{1}{\varepsilon} \zeta\left(\frac{2 \phi}{\varepsilon}\right) \neq|\nabla \phi| \frac{1}{\varepsilon} \zeta\left(\frac{\phi}{\varepsilon}\right)$ ). This is the origin of some numerical subtleties in the Level Set method. We will see how redistancing and renormalization give answer to this problem.

### 2.5.2 General level set formulation of a gradient based active contour method

Thanks to the above approximation, an Eulerian expression of the (approximated) active contour energy (5) can be written as

$$
F_{\varepsilon}(\phi)=\int_{\Omega} g(|\nabla \mathbb{I}|) \frac{1}{\varepsilon} \zeta\left(\frac{\phi}{\varepsilon}\right)|\nabla \phi| d x
$$

Let us compute, independently of the dimension, the functional derivative of this energy. Using the transport equation verified by $\phi$ and equation verified by $|\nabla \phi|$,

$$
\begin{aligned}
& \frac{d}{d t} F_{\varepsilon}(\phi)=\int_{\Omega} g(|\nabla \mathbb{I}|) {\left[\frac{1}{\varepsilon^{2}} \zeta^{\prime}\left(\frac{\phi}{\varepsilon}\right) \partial_{t} \phi|\nabla \phi|+\frac{1}{\varepsilon} \zeta\left(\frac{\phi}{\varepsilon}\right) \partial_{t}|\nabla \phi|\right] d x } \\
&=\int_{\Omega} g(|\nabla \mathbb{I}|)\left[\frac{1}{\varepsilon^{2}} \zeta^{\prime}\left(\frac{\phi}{\varepsilon}\right)(-u \cdot \nabla \phi)|\nabla \phi|+\frac{1}{\varepsilon} \zeta\left(\frac{\phi}{\varepsilon}\right) \frac{\partial_{t} \nabla \phi \cdot \nabla \phi}{|\nabla \phi|}\right] d x \\
&=\int_{\Omega} g(|\nabla \mathbb{I}|)\left[-u \cdot \nabla\left(\frac{1}{\varepsilon} \zeta\left(\frac{\phi}{\varepsilon}\right)\right)|\nabla \phi|\right]-\operatorname{div}\left(g(|\nabla \mathbb{I}|) \frac{\nabla \phi}{|\nabla \phi|} \frac{1}{\varepsilon} \zeta\left(\frac{\phi}{\varepsilon}\right)\right) \partial_{t} \phi d x
\end{aligned}
$$

where we performed an integration by parts to get the last expression. Still using the transport equation, $\partial_{t} \phi=-u \cdot \nabla \phi$ so that

$$
\begin{aligned}
\frac{d}{d t} F_{\varepsilon}(\phi)=\int_{\Omega} g(|\nabla \mathbb{I}|)[-u & \left.\cdot \nabla\left(\frac{1}{\varepsilon} \zeta\left(\frac{\phi}{\varepsilon}\right)\right)|\nabla \phi|+|\nabla \phi| \nabla\left(\frac{1}{\varepsilon} \zeta\left(\frac{\phi}{\varepsilon}\right)\right) \cdot u\right] \\
& +g(|\nabla \mathbb{I}|) \operatorname{div}\left(\frac{\nabla \phi}{|\nabla \phi|}\right) \frac{1}{\varepsilon} \zeta\left(\frac{\phi}{\varepsilon}\right) \nabla \phi \cdot u+\nabla[g(|\nabla \mathbb{I}|)] \cdot \frac{\nabla \phi}{|\nabla \phi|} \frac{1}{\varepsilon} \zeta\left(\frac{\phi}{\varepsilon}\right) \nabla \phi \cdot u d x
\end{aligned}
$$

Finally, we got

$$
\frac{d}{d t} F_{\varepsilon}(\phi)=\int_{\Omega}\left[g(|\nabla \mathbb{I}|) \operatorname{div}\left(\frac{\nabla \phi}{|\nabla \phi|}\right)+\nabla[g(|\nabla \mathbb{I}|)] \cdot \frac{\nabla \phi}{|\nabla \phi|}\right] \frac{\nabla \phi}{|\nabla \phi|} \cdot u|\nabla \phi| \frac{1}{\varepsilon} \zeta\left(\frac{\phi}{\varepsilon}\right) d x
$$

which is the level translation of (6), up to the approximation of the surface measure given by $|\nabla \phi| \frac{1}{\varepsilon} \zeta\left(\frac{\phi}{\varepsilon}\right) d x$.

Remark 3 In the case where curvature energy has to be taken into account, the level-set formalism can of course be used and is also dimension independent. Actually the curvature energy can take the form

$$
\mathcal{G}_{\varepsilon}(\phi)=\int_{\Omega} G(\kappa(\phi))|\nabla \phi| \frac{1}{\varepsilon} \zeta\left(\frac{\phi}{\varepsilon}\right) d x
$$

where the more standard case corresponds to $G(r)=\frac{1}{2} r^{2}$. As above, the time derivative of this curvature energy corresponds to the power of curvature forces $H_{\varepsilon}(x, t)$, since we have

$$
\begin{equation*}
\frac{d}{d t} \mathcal{G}_{\varepsilon}(\phi)=d \mathcal{G}_{\varepsilon}(\phi)\left(\partial_{t} \phi\right)=d \mathcal{G}_{\varepsilon}(\phi)(-u \cdot \nabla \phi)=-\int_{\Omega} H_{\varepsilon}(x, t) \cdot u d x \tag{8}
\end{equation*}
$$

After a lengthy computation, we obtain:

$$
H_{\varepsilon}(x, t)=\operatorname{div}\left[-G(\kappa(\phi)) \frac{\nabla \phi}{|\nabla \phi|}+\frac{1}{|\nabla \phi|} \mathbb{P}_{\nabla \phi^{\perp}}\left(\nabla\left[|\nabla \phi| G^{\prime}(\kappa(\phi))\right]\right)\right] \frac{1}{\varepsilon} \zeta\left(\frac{\phi}{\varepsilon}\right) \nabla \phi .
$$

### 2.6 Region-based methods

### 2.6.1 Idea

In the above methods, one drawback is that the image gradient is involved, which could be a problem for noisy images. Other methods are built on the principle of capturing regions instead of capturing contours delimiting them. The pioneering method of Chan \& Vese (2001) is based on the minimization of the so-called Mumford-Shah functional:

$$
E^{M S}(u, \Gamma)=\sum_{i} \int_{\Omega_{i}}|u-\mathbb{I}|^{2} d x+\mu|\Gamma|+\nu \int_{\Omega \backslash \Gamma}|\nabla u|^{2} d x .
$$

A minimizer of such an energy is therefore regular outside $\Gamma$, and try fit the image on the regions $\Omega_{i}$, while minimizing the length of $\Gamma$. Starting from that energy, Chan \& Vese proposed the following level-set formulation by looking for a minimizer in the class of piecewise constant functions, therefore canceling the last term:

$$
E^{C V}\left(\phi, c_{1}, c_{2}\right)=\int_{\Omega}\left|c_{1}-\mathbb{I}\right|^{2} H\left(\frac{\phi}{\varepsilon}\right)+\left|c_{2}-\mathbb{I}\right|^{2}\left(1-H\left(\frac{\phi}{\varepsilon}\right)\right) d x+\mu \int_{\Omega}|\nabla \phi| \frac{1}{\varepsilon} \zeta\left(\frac{\phi}{\varepsilon}\right) d x
$$

First note that deriving with respect to $c_{i}$ immediately gives that at optimum,

$$
c_{1}=\frac{\int_{\Omega} \mathbb{I}(x) H\left(\frac{\phi}{\varepsilon}\right) d x}{\int_{\Omega} H\left(\frac{\phi}{\varepsilon}\right)}, \quad c_{2}=\frac{\int_{\Omega} \mathbb{I}(x)\left(1-H\left(\frac{\phi}{\varepsilon}\right)\right) d x}{\int_{\Omega} 1-H\left(\frac{\phi}{\varepsilon}\right)}
$$

and the gradient flow associated to this energy can be obtained by:

$$
\partial_{t} \phi=\left[\mu \kappa(\phi)-\left(\mathbb{I}-c_{1}\right)^{2}+\left(\mathbb{I}-c_{2}\right)^{2}\right]|\nabla \phi| \frac{1}{\varepsilon} \zeta\left(\frac{\phi}{\varepsilon}\right)
$$

with $c_{1}, c_{2}$ obtained as above. Typically we take a function $\frac{1}{\varepsilon} \zeta\left(\frac{\phi}{\varepsilon}\right)>0$ to allow creation of new regions.
Some generalizations are the following: first we can seek unknown patterns instead of simple constants. This amounts to consider functions instead of constants. We define

$$
\begin{aligned}
E^{C V}\left(\phi, u_{1}, u_{2}\right)=\int_{\Omega}\left|u_{1}(x)-\mathbb{I}\right|^{2} H\left(\frac{\phi}{\varepsilon}\right)+\mid u_{2}(x) & -\left.\mathbb{I}\right|^{2}\left(1-H\left(\frac{\phi}{\varepsilon}\right)\right) d x+\mu \int_{\Omega}|\nabla \phi| \frac{1}{\varepsilon} \zeta\left(\frac{\phi}{\varepsilon}\right) d x \\
& +\nu \int_{\Omega}\left|\nabla u_{1}\right|^{2} H\left(\frac{\phi}{\varepsilon}\right) d x+\nu \int_{\Omega}\left|\nabla u_{2}\right|^{2}\left(1-H\left(\frac{\phi}{\varepsilon}\right)\right) d x .
\end{aligned}
$$

which gives as Euler-Lagrange equations:

$$
u_{2}-\mathbb{I}=\nu \Delta u_{2} \text { on } \phi>0, \quad u_{1}-\mathbb{I}=\nu \Delta u_{1} \text { on } \phi<0
$$

with homogeneous Neumann boundary condition on $\phi=0$ for the $u_{i}$. The the descent algorithm gives:

$$
\partial_{t} \phi=\left[\mu \kappa(\phi)-\nu-\left(\mathbb{I}-u_{1}\right)^{2}+\left(\mathbb{I}-u_{2}\right)^{2}-\nu\left|\nabla u_{2}\right|^{2}+\nu\left|\nabla u_{1}\right|^{2}\right]|\nabla \phi| \frac{1}{\varepsilon} \zeta\left(\frac{\phi}{\varepsilon}\right)
$$

Another extension is to capture more than two regions. Does this need to introduce as much level-set functions as regions? Hopefully not, due to the four colors theorem. This states that each partition of $\mathbb{R}^{2}$ in regions can be colored with 4 colors, so that to contiguous regions are not painted with the same color. This can be implemented with two level-set functions using the combination of signs of those two functions. We therefore introduce

$$
\begin{aligned}
& E\left(\phi_{1}, \phi_{2}, u_{11}, u_{12}, u_{21}, u_{22}\right)=\int_{\Omega}\left|u_{11}(x)-\mathbb{I}\right|^{2} H\left(\frac{\phi_{1}}{\varepsilon}\right) H\left(\frac{\phi_{2}}{\varepsilon}\right)+\left|u_{12}(x)-\mathbb{I}\right|^{2}\left(1-H\left(\frac{\phi_{1}}{\varepsilon}\right)\right) H\left(\frac{\phi_{2}}{\varepsilon}\right) \\
& +\left|u_{21}(x)-\mathbb{I}\right|^{2}\left(1-H\left(\frac{\phi_{2}}{\varepsilon}\right)\right) H\left(\frac{\phi_{1}}{\varepsilon}\right)+\left|u_{22}(x)-\mathbb{I}\right|^{2}\left(1-H\left(\frac{\phi_{1}}{\varepsilon}\right)\right)\left(1-H\left(\frac{\phi_{2}}{\varepsilon}\right)\right) d x \\
& +\nu \int_{\Omega}\left|\nabla u_{11}\right|^{2} H\left(\frac{\phi_{1}}{\varepsilon}\right) H\left(\frac{\phi_{2}}{\varepsilon}\right)+\left|\nabla u_{12}\right|^{2}\left(1-H\left(\frac{\phi_{1}}{\varepsilon}\right)\right) H\left(\frac{\phi_{2}}{\varepsilon}\right)+\left|\nabla u_{21}\right|^{2}\left(1-H\left(\frac{\phi_{2}}{\varepsilon}\right)\right) H\left(\frac{\phi_{1}}{\varepsilon}\right) \\
& +\left|\nabla u_{22}\right|^{2}\left(1-H\left(\frac{\phi_{1}}{\varepsilon}\right)\right)\left(1-H\left(\frac{\phi_{2}}{\varepsilon}\right)\right) d x+\mu \int_{\Omega}\left|\nabla \phi_{1}\right| \frac{1}{\varepsilon} \zeta\left(\frac{\phi_{1}}{\varepsilon}\right)+\left|\nabla \phi_{2}\right| \frac{1}{\varepsilon} \zeta\left(\frac{\phi_{2}}{\varepsilon}\right) d x
\end{aligned}
$$

and we generalize the above descent algorithm.
Exercise 1 Compute the gradient flow associated to that energy and write the active contour algorithm associated to it.

### 2.6.2 Tests

In figure 6, the Chan-Vese algorithm shows its ability to reach a steady state in the two-objects noisy test case. In the more complicated three objects noisy case, which is a difficult test case for contouring method, it perform also well. However, when it comes to contour regions where the image intensity is not constant, the Chan-Vese algorithm can be fooled (see figure 7 where two regions of are localized by assuming a mean value, giving as bad segmentation. To localize region with possibly varying intensity, the extension to patterns presented above is a better choice. This is even more visible on a nosier image which depicts a spiral. The original Chan-Vese algorithm find two regions which ignore the natural segmentation, while Li algorithm behaves quite well (figure 8).


FIGURE 4: Region-based contouring of (Chan-Vese, 2001). The two-objects and three-objects cases.


FIGURE 5: Region-based contouring of (Chan-Vese, 2001) / Extension to the pattern matching case (Li et al).


FIGURE 6: Region-based contouring of (Chan-Vese, 2001) / Extension to the pattern matching case (Li et al).


[^0]:    ${ }^{1}$ It is possible to remove this assumption with the addition of another auxiliary function.

