Designing proof systems from programming features: states and exceptions considered as dual effects

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July 5., 2011 – PPS – Groupe de Travail Sémantique

Outline

The Curry Howard Lambek correspondence

intuitionistic logic	typed lambda calculus	cartesian closed categories
propositions	types	objects
proofs	terms	morphisms

What about non-functional features in programming languages?
i.e., what about computational effects?

Claim. Each computational effect has an associated logic

In this talk: The effects of states and exceptions, with their logics

A surprising result

There is a symmetry between the logics for states and exceptions, based on the well-known categorical duality:

for states	for exceptions
$X \mapsto X \times S$ with fixed S	$X \mapsto X + E$ with fixed E

Outline

1. A symmetry between states and exceptions
   at the semantics level

2. A symmetry between states and exceptions
   at the logical level

3. About "decorated" proofs

Reference:
J.-G. Dumas, D. Duval, L. Fousse, J.-C. Reynaud
States and exceptions considered as dual effects
`http://arxiv.org/abs/1001.1662 (v4)`

Outline

Exceptions: values

When dealing with exceptions, there are two kinds of values:

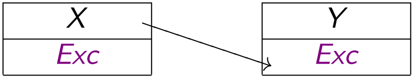
- non-exceptional values
- exceptions

$$X + \mathit{Exc} \;=\; \begin{array}{|c|}\hline X \\ \hline \mathit{Exc} \\ \hline \end{array}$$

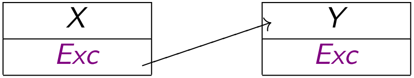
Exceptions: functions

$$f : X + Exc \to Y + Exc$$

- f throws an exception if it may map a non-exceptional value to an exception



- f catches an exception if it may map an exception to a non-exceptional value



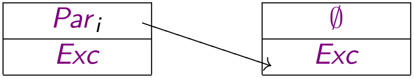
Exceptions: the KEY THROW operations

Exc = set of exceptions
$ExCstr$ = set of exception constructors (or exception types)

For each $i \in ExCstr$:

- Par_i = set of parameters
- $t_i : Par_i \to Exc$ = the KEY THROW operations

  or $t_i : Par_i + Exc \to Exc$ such that $\forall\, e \in Exc,\ t_i(e) = e$



– t_i throws exceptions of constructor i

– t_i propagates exceptions

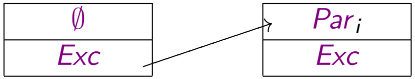
E.g. $Exc = \sum_i Par_i$ with the t_i's as coprojections

Exceptions: the KEY CATCH operations

For each $i \in ExCstr$:

- $c_i : Exc \to Par_i + Exc$ = the KEY CATCH operations

$$\forall\, p \in Par_i \quad \begin{cases} c_i(t_i(p)) = p \in Par_i \subseteq Par_i + Exc \\ c_i(t_j(p)) = t_j(p) \in Exc \subseteq Par_i + Exc \quad (\forall\, j \neq i) \end{cases}$$



  - c_i catches exceptions of constructor i
  - c_i propagates exceptions of constructor $j \neq i$

E.g. $Exc = \sum_i Par_i$ with the t_i's as coprojections:
these equations define the c_i's

Exceptions: encapsulation

The key throwing and catching operations are encapsulated for building the usual raising and handling constructions

- The usual raising construction throws an exception viewed as an element of some type X
- The usual handling construction catches an exception inside a block carefully delimitated

Exceptions: the RAISE (or THROW) construction

The usual raising construction throws an exception viewed as an element of some type X

- From key throwing (t_i)
  to raising ($raise_{i,Y}$ or $throw_{i,Y}$):

$$raise_{i,Y}(a) = t_i(a) \in Y + Exc$$

$$\begin{tikzcd} Par_i \arrow[r, "raise_{i,Y}"] \arrow[dr, "t_i"'] \arrow[dr, phantom, near end, shift left=3, "="] & Y + Exc \\ & Exc \arrow[u, "\subseteq"'] \end{tikzcd}$$

Exceptions: the HANDLE (or TRY...CATCH) construction

The usual handling construction catches an exception
inside a block carefully deliminated

- From key catching (c_i)
  to catching ($catch\, i\, \{g\}$):

$$\begin{tikzcd}
 & Par_i \arrow[d, "\subseteq"'] \arrow[rd, "g"] \arrow[rd, phantom, shift right=3ex, "="] & \\
Exc \arrow[r, "c_i"] & Par_i + Exc \arrow[r] & Y + Exc \\
 & Exc \arrow[u, "\subseteq"] \arrow[ru, "\subseteq"'] \arrow[ru, phantom, shift left=3ex, "="] &
\end{tikzcd}$$

- From catching ($catch\, i\, \{g\}$)
  to handling ($f\ handle\ i \Rightarrow g$ or $try\ \{f\} catch\, i\, \{g\}$):

$$\begin{tikzcd}
 & Y \arrow[d, "\subseteq"'] \arrow[rd, "\subseteq"] \arrow[rd, phantom, shift right=3ex, "="] & \\
X \arrow[r, "f"] & Y + Exc \arrow[r] & Y + Exc \\
 & Exc \arrow[u, "\subseteq"] \arrow[ru, "catch\, i\, \{g\}"'] \arrow[ru, phantom, shift left=3ex, "="] &
\end{tikzcd}$$

States

St = set of states
Loc = set of locations

For each $i \in Loc$:

- Val_i = set of values
- $l_i : St \to Val_i$ = lookup function

  or $l_i : St \to Val_i \times St$ such that $\forall s \in St,\ l_i(s) = (-, s)$
- $u_i : Val_i \times St \to St$ = update function

$$\forall\, v \in Val_i \ \ \forall\, s \in St \quad \begin{cases} l_i(u_i(v,s)) = v \\ l_j(u_i(v,s)) = l_j(s) \ \ (\forall\, j \neq i) \end{cases}$$

E.g. $St = \prod_i Val_i$ with the l_i's as projections:
these equations define the u_i's

Duality of semantics

States	**Exceptions**
$i \in Loc,\ Val_i$ $St\ (= \prod_{i \in Loc} Val_i)$	$i \in ExCstr,\ Par_i$ $Exc\ (= \sum_{i \in ExCstr} Par_i)$
$l_i : St \to Val_i$ $u_i : Val_i \times St \to St$	$Exc \leftarrow Par_i : t_i$ $Par_i + Exc \leftarrow Exc : c_i$
$$\begin{tikzcd} Val_i \times St \arrow[r, "pr"] \arrow[d, "u_i"'] \arrow[dr, phantom, "="] & Val_i \arrow[d, "id"] \\ St \arrow[r, "l_i"] & Val_i \end{tikzcd}$$ $$\begin{tikzcd} Val_i \times St \arrow[r, "pr"] \arrow[d, "u_i"'] & St \arrow[r, "l_j"] \arrow[d, phantom, "="] & Val_j \arrow[d, "id"] \\ St \arrow[rr, "l_j"] & & Val_j \end{tikzcd}$$	$$\begin{tikzcd} Par_i + Exc \arrow[dr, phantom, "="] & Par_i \arrow[l, "in"'] \\ Exc \arrow[u, "c_i"] & Par_i \arrow[l, "t_i"] \arrow[u, "id"'] \end{tikzcd}$$ $$\begin{tikzcd} Par_i + Exc & Exc \arrow[l, "in"'] \arrow[d, phantom, "="] & Par_j \arrow[l, "t_j"'] \\ Exc \arrow[u, "c_i"] & & Par_j \arrow[ll, "t_j"] \arrow[u, "id"'] \end{tikzcd}$$

- So, there is a duality between states and exceptions, at the semantics level, involving a set of states St and a set of exceptions Exc.
- But states and exceptions are computational effects: the "type of states" and the "type of exceptions" are hidden, they do not appear explicitly in the syntax.
- In fact, the duality at the semantics level comes from a duality of states and exceptions seen as computational effects, at the logical level.

Outline

Monads for effects

[Moggi 1991] *The basic idea behind the categorical semantics of effects is that we distinguish the object X of values from the object TX of computations (for some endofunctor T)*

Programs of type Y with a parameter of type X ought to be interpreted by morphisms with codomain TY, but for their domain there are two alternatives, either X or TX.

1. Moggi chooses the first alternative:

   a program $X \to Y$ is interpreted by a morphism $X \to TY$

   Then T must be a monad – for substitution
   with a strength – for the context
2. The second alternative would be:

   a program $X \to Y$ is interpreted by a morphism $TX \to TY$

Monads for effects: exceptions

The monad of exceptions is $TX = X + \mathit{Exc}$.

1. First alternative.
   A program of type Y with a parameter of type X
   is interpreted by a morphism $X \to Y + \mathit{Exc}$.
   $\Longrightarrow$ it may throw an exception
   $\Longrightarrow$ it cannot catch an exception
2. Second alternative.
   A program of type Y with a parameter of type X
   is interpreted by a morphism $X + \mathit{Exc} \to Y + \mathit{Exc}$.
   $\Longrightarrow$ it may throw an exception
   $\Longrightarrow$ it may catch an exception

Effects, more generally

Claim. A computational effect is

an apparent lack of soundness

There is a computational effect when:

- at first sight, the intended semantics is not a model of the syntax
- but the syntax may be "decorated" so as to recover soundness

The monads approach from this point of view:

- operations are decorated as values or computations and every value can be seen as a computation
- a computation $f^c : X \to Y$ stands for $f : X \to TY$
- a value $f^v : X \to Y$ stands for $f : X \to Y \xrightarrow{\eta_Y} TY$

States, apparently

The intended semantics (one location):

$$\begin{cases} l : St \to Val \\ u : Val \times St \to St \\ \forall v \in Val \;\; \forall s \in St \;\; l(u(v,s)) = v \end{cases}$$

IS NOT a model of the apparent syntax

Apparent
$l : \mathbb{1} \to V$
$u : V \to \mathbb{1}$
$l \circ u = id : V \to V$

States, explicitly

The intended semantics (one location)

$$\begin{cases} l : St \to Val \\ u : Val \times St \to St \\ \forall\, v \in Val \;\; \forall\, s \in St \;\; l(u(v,s)) = v \end{cases}$$

IS a model of the explicit syntax

Explicit
$l : S \to V$
$u : V \times S \to S$
$l \circ u = pr : V \times S \to V$

States, equationally

There are two equational logics “for states”:

Apparent logic
NOT sound
close to the syntax

Explicit logic
sound
FAR from the syntax

Claim. There is a third logic for states – NOT “truly” equational:

Decorated logic
sound
close to the syntax

States as effect: decorations

The apparent syntax may be decorated:

- An operation $f : X \to Y$ is decorated as
  $f^{(0)} : X \to Y$ if f is pure
  $f^{(1)} : X \to Y$ if f is an accessor (cf. `const` methods in `C++`)
  $f^{(2)} : X \to Y$ if f is a modifier
- An equation $f = g$ is decorated as
  $f =^{(sg)} g$ (strong) if f and g coincide on results and on states
  $f =^{(wk)} g$ (weak) if f and g coincide on results (only)

Apparent
$l : \mathbb{1} \to V$
$u : V \to \mathbb{1}$
$l \circ u = id_V : V \to V$

Decorated
$l^{(1)} : \mathbb{1} \to V$
$u^{(2)} : V \to \mathbb{1}$
$l \circ u =^{(wk)} id_V : V \to V$

States as effect: expliciting the decorations

The decorated syntax may be explicited

- For operations:

  $f^{(0)} : X \to Y$ as $f : X \to Y$

  $f^{(1)} : X \to Y$ as $f : X \times S \to Y$

  $f^{(2)} : X \to Y$ as $f : X \times S \to Y \times S$

- For equations:

  $f =^{(sg)} g$ as $f = g : X \times S \to Y \times S$

  $f =^{(wk)} g$ as $pr_Y \circ f = pr_Y \circ g : X \times S \to Y$

Decorated
$l^{(1)} : \mathbb{1} \to V$
$u^{(2)} : V \to \mathbb{1}$
$l \circ u =^{(wk)} id_V : V \times S \to V$

Explicit
$l : \mathbb{1} \times S \to V$
$u : V \times S \to S$
$l \circ u = pr_V : V \times S \to V$

States as effect: three logics

$$\begin{tikzcd}
& \boxed{\begin{array}{l}\text{Decorated}\\ \hline l^{(1)} : \mathbb{1} \to V \\ u^{(2)} : V \to \mathbb{1} \\ l \circ u =^{(wk)} id_V\end{array}} \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
\boxed{\begin{array}{l}\text{Apparent}\\ \hline l : \mathbb{1} \to V \\ u : V \to \mathbb{1} \\ l \circ u = id_V\end{array}} & & \boxed{\begin{array}{l}\text{Explicit}\\ \hline l : S \to V \\ u : V \times S \to S \\ l \circ u = pr_V\end{array}}
\end{tikzcd}$$

The intended semantics

- IS NOT a model of the apparent syntax (effect)
- IS a model of the explicit syntax (obviously)
- IS a model of the decorated syntax (by adjunction)

Exceptions as effect

The intended semantics (one exception constructor):

$$\begin{cases} t : Par \to Exc \\ c : Exc \to Par + Exc \\ \forall\, p \in Par \;\; c(t(p)) = p \end{cases}$$

IS NOT a model of the apparent syntax
IS a model of the explicit syntax

Apparent
$t : P \to \mathbb{0}$
$c : \mathbb{0} \to P$
$c \circ t = id : P \to P$

Explicit
$t : P \to E$
$c : E \to P + E$
$c \circ t = in : P \to P + E$

Exceptions as effect: decorations

The apparent syntax may be decorated:

- An operation $f : X \to Y$ is decorated as
  $f^{(0)} : X \to Y$ if f is pure
  $f^{(1)} : X \to Y$ if f is a propagator (it may throw exceptions)
  $f^{(2)} : X \to Y$ if f is a catcher (it may throw and catch exc.)
- An equation $f = g$ is decorated as
  $f =^{(sg)} g$ (strong) if f and g coincide on exc. and on values
  $f =^{(wk)} g$ (weak) if f and g coincide on values (only)

Apparent
$t : P \to \mathbb{0}$
$c : \mathbb{0} \to P$
$c \circ t = id : P \to P$

Decorated
$t^{(1)} : P \to \mathbb{0}$
$c^{(2)} : \mathbb{0} \to P$
$c^{(2)} \circ t^{(1)} =^{(wk)} id^{(0)} : P \to P$

Exceptions as effect: expliciting the decorations

The decorated syntax may be explicited

- For operations:

  $f^{(0)} : X \to Y$ as $f : X \to Y$

  $f^{(1)} : X \to Y$ as $f : X \to Y + E$

  $f^{(2)} : X \to Y$ as $f : X + E \to Y + E$

- For equations:

  $f =^{(sg)} g$ as $f = g : X \times S \to Y \times S$

  $f =^{(wk)} g$ as $f \circ in_X = g \circ in_X : X \to Y + E$

Decorated
$t^{(1)} : P \to \mathbb{0}$
$c^{(2)} : \mathbb{0} \to P$
$c^{(2)} \circ t^{(1)} =^{(wk)} id^{(0)} : P \to P$

Explicit
$t : P \to E$
$c : E \to P + E$
$c \circ t = in : P \to P + E$

Exceptions as effect: three logics

$$\begin{tikzcd}
& \boxed{\begin{array}{l}\text{Decorated}\\ \hline t^{(1)} : P \to \mathbb{0} \\ c^{(2)} : \mathbb{0} \to P \\ c \circ t =^{(wk)} id_P\end{array}} \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
\boxed{\begin{array}{l}\text{Apparent}\\ \hline t : P \to \mathbb{0} \\ c : \mathbb{0} \to P \\ c \circ t = id_P\end{array}} & & \boxed{\begin{array}{l}\text{Explicit}\\ \hline t : P \to E \\ c : E \to P + E \\ c \circ t = in_P\end{array}}
\end{tikzcd}$$

The intended semantics

- IS NOT a model of the apparent syntax (effect)
- IS a model of the explicit syntax (obviously)
- IS a model of the decorated syntax (by adjunction)

Duality of effects

States	**Exceptions**
$i \in Loc,\ V_i$ $\mathbb{1}$	$i \in ExCstr,\ P_i$ $\mathbb{0}$
$l_i^{(1)} : \mathbb{1} \to V_i$ $u_i^{(2)} : V_i \to \mathbb{1}$	$\mathbb{0} \leftarrow P_i : t_i^{(1)}$ $P_i \leftarrow \mathbb{0} : c_i^{(2)}$
$$\begin{tikzcd} V_i \arrow[r, "id"] \arrow[d, "u_i"'] \arrow[dr, phantom, "=^{(wk)}"] & V_i \arrow[d, "id"] \\ \mathbb{1} \arrow[r, "l_i"] & V_i \end{tikzcd}$$ $$\begin{tikzcd} V_i \arrow[r] \arrow[d, "u_i"'] & \mathbb{1} \arrow[r, "l_j"] \arrow[d, phantom, "=^{(wk)}"] & V_j \arrow[d, "id"] \\ \mathbb{1} \arrow[rr, "l_j"] & & V_j \end{tikzcd}$$	$$\begin{tikzcd} P_i \arrow[dr, phantom, "=^{(wk)}"] & P_i \arrow[l, "id"'] \\ \mathbb{0} \arrow[u, "c_i"] & P_i \arrow[l, "t_i"'] \arrow[u, "id"'] \end{tikzcd}$$ $$\begin{tikzcd} P_i & \mathbb{0} \arrow[l] \arrow[d, phantom, "=^{(wk)}"] & P_j \arrow[l, "t_j"'] \\ \mathbb{0} \arrow[u, "c_i"] & & P_j \arrow[ll, "t_j"'] \arrow[u, "id"'] \end{tikzcd}$$

Outline

Operations and equations

- The monads approach leads to Lawvere theories
  for getting operations and equations [Plotkin&Power 2001]
  This can be extended
  - with exception monads [Schroeder&Mossakowski 2004]
  - with coalgebras [Levy 2006]
  - with handlers [Plotkin&Pretnar 2009]

  Then
  - `lookup`, `update`, `raise` are algebraic operations
  - `handle` IS NOT an algebraic operation

- Our approach generalizes algebraic specifications

  it involves (decorated) operations and equations

  Then
  - `catching` exceptions is symmetric to `updating` states

A framework for effects

A language without effects is defined with respect to one logic

$$L$$

A language with effects is defined with respect to a span of logics

$$\begin{tikzcd} & L_{\mathrm{deco}} \arrow[dl] \arrow[dr] & \\ L_{\mathrm{app}} & & L_{\mathrm{expl}} \end{tikzcd}$$

Morphisms of logics are defined in the category of diagrammatic logics [Duval&Lair 2002]. This is based on:

- Adjunctions [Kan 1958]
- Categories of fractions [Gabriel&Zisman 1967]
- Limit sketches [Ehresmann 1968]

One logic: models

A diagrammatic logic is a left adjoint functor L with a full and faithful right adjoint R

$$\begin{tikzcd} \mathbf{S} \arrow[rr, "L"] \arrow[rr, phantom, shift right=3, "\perp"] & & \mathbf{T} \arrow[ll, bend left, "R \text{ (f.f.)}"] \end{tikzcd}$$

induced by a morphism of limit sketches

- $\mathbf{S}$ is the category of specifications
- $\mathbf{T}$ is the category of theories
- Each specification Σ presents the theory $L\Sigma$
- A model $M : \Sigma \to \Theta$ is an "oblique" morphism:
  $M \colon L\Sigma \to \Theta$ in $\mathbf{T}$ or $M \colon \Sigma \to R\Theta$ in $\mathbf{S}$

One logic: proofs

T is a category of fractions on **S**:
a fraction is a cospan in **S** with numerator σ
and denominator τ such that $L\tau$ is invertible in **T**

$$\begin{tikzcd}
\Sigma_1 \arrow[r, "\sigma"] & \Sigma'_2 \arrow[r, dashed, shift right] & \Sigma_2 \arrow[l, "\tau"', shift right]
\end{tikzcd}$$

This fraction can be seen as

- an instance of the specification Σ_1 in Σ_2
- or an inference rule with hypothesis Σ_2 and conclusion Σ_1

The inference step is the composition of fractions:
applying a rule with hypothesis H and conclusion C
to an instance of H in Σ
yields an instance of C in Σ.

A category of logics

A morphism of logics $F\colon L_1 \to L_2$
is a pair of left adjoint functors (F_S, F_T)
in a commutative square

$$\begin{tikzcd}
\mathbf{S}_1 \arrow[r, "L_1"] \arrow[d, "F_S"'] \arrow[dr, phantom, "\cong"] & \mathbf{T}_1 \arrow[d, "F_T"] \\
\mathbf{S}_2 \arrow[r, "L_2"'] & \mathbf{T}_2
\end{tikzcd}$$

induced by a commutative square of limit sketches

This yields the category of diagrammatic logics

Decorated proofs

$$\begin{tikzcd} & L_{\mathrm{deco}} \arrow[dl, "F_{\mathrm{app}}"'] \arrow[dr, "F_{\mathrm{expl}}"] & \\ L_{\mathrm{app}} & & L_{\mathrm{expl}} \end{tikzcd}$$

In this talk, for states and exceptions,
L_{app} and L_{expl} are (variants of) equational logic.

Each decorated proof is mapped to an equational proof

- either by dropping the decorations (by F_{app})
  $\rightarrow$ an "uninteresting" proof
- or by expliciting the decorations (by F_{expl})
  $\rightarrow$ a "complicated" proof

Some decorated rules for states (1)

$$\boxed{\begin{array}{ll} (0\text{-to-}1) & \dfrac{f^{(0)}}{f^{(1)}} \\[2ex] (1\text{-to-}2) & \dfrac{f^{(1)}}{f^{(2)}} \end{array}}$$

$$\boxed{\begin{array}{ll} (sg\text{-subs}) & \dfrac{g_1^{(2)} =^{(sg)} g_2^{(2)}}{(g_1 \circ f)^{(2)} =^{(sg)} (g_2 \circ f)^{(2)}} \\[2ex] (sg\text{-repl}) & \dfrac{f_1^{(2)} =^{(sg)} f_2^{(2)}}{(g \circ f_1)^{(2)} =^{(sg)} (g \circ f_2)^{(2)}} \end{array}}$$

$$\boxed{\begin{array}{ll} (wk\text{-subs}) & \dfrac{g_1^{(2)} =^{(wk)} g_2^{(2)}}{(g_1 \circ f)^{(2)} =^{(wk)} (g_2 \circ f)^{(2)}} \\[2ex] (wk\text{-repl}) & \dfrac{f_1^{(2)} =^{(wk)} f_2^{(2)} \quad g^{(0)}}{(g \circ f_1)^{(2)} =^{(wk)} (g \circ f_2)^{(2)}} \end{array}}$$

Some decorated rules for states (2)

$$\boxed{\begin{array}{ll} (sg\text{-to-}wk) & \dfrac{f^{(2)} =^{(sg)} g^{(2)}}{f^{(2)} =^{(wk)} g^{(2)}} \\ (wk\text{-to-}sg) & \dfrac{f^{(1)} =^{(wk)} g^{(1)}}{f^{(1)} =^{(sg)} g^{(1)}} \end{array}}$$

and the `lookup`'s form a "decorated product" $(l_j^{(1)})_{j \in Loc}$ such that

$$f^{(2)} =^{(sg)} g^{(2)} \iff \forall j \in Loc,\ (l_j \circ f)^{(2)} =^{(wk)} (l_j \circ g)^{(2)}$$

$$\begin{tikzcd} & & V_i \\ X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathbb{1} \arrow[ru, "l_i"] \arrow[rd, "l_j"'] \arrow[r, phantom, "\vdots"] & \vdots \\ & & V_j \end{tikzcd}$$

A decorated proof (for states)

Proposition. For every $i \in Loc$:

- Semantically: $\forall s \in St,\ u_i(l_i(s), s) = s$
- Explicitly: $u_i \circ l_i = id_S$
- Decorated: $u_i^{(2)} \circ l_i^{(1)} =^{(sg)} id_{\mathbb{1}}^{(0)}$

Proof. $\forall j \in Loc,\ l_j^{(1)} \circ u_i^{(2)} \circ l_i^{(1)} =^{(wk)} l_j^{(1)}$

When $j = i$:

$$(wk\text{-subs})\ \dfrac{l_i \circ u_i =^{(wk)} id_{V_i}}{l_i \circ u_i \circ l_i =^{(wk)} l_i}$$

When $j \neq i$:

$$(wk\text{-trans})\ \dfrac{(wk\text{-subs})\ \dfrac{l_j \circ u_i =^{(wk)} l_j \circ \langle\,\rangle_{V_i}}{l_j \circ u_i \circ l_i =^{(wk)} l_j \circ \langle\,\rangle_{V_i} \circ l_i} \qquad \begin{array}{l} (sg\text{-repl}) \\ (sg\text{-to-}wk) \end{array} \dfrac{\dfrac{\dfrac{\vdots}{\langle\,\rangle_{V_i} \circ l_i =^{(sg)} id_{\mathbb{1}}}}{l_j \circ \langle\,\rangle_{V_i} \circ l_i =^{(sg)} l_j}}{l_j \circ \langle\,\rangle_{V_i} \circ l_i =^{(wk)} l_j}}{l_j \circ u_i \circ l_i =^{(wk)} l_j}$$

Decorated rules and proofs (for exceptions)

Decorated rules and proofs for exceptions
are dual to decorated rules and proofs for states.

Proposition. For every $i \in ExCstr$:

- Semantically: $\forall e \in Exc,\ t_i(c_i(e)) = e$
- Explicitly: $t_i \circ c_i = id_E$
- Decorated: $t_i^{(1)} \circ c_i^{(2)} =^{(sg)} id_{\mathbb{1}}^{(0)}$

Proof. Dual to the proof for states.

More decorated proofs (for states)

Equations from [Plotkin&Power 2002] as stated in [Melliès 2010]

- *Interaction update-update:*
  storing a value v and then a value v' at the same location i is just like storing the value v' in the location i. $\forall i \in Loc$,

$$u_i^{(2)} \circ (u_i \times id_{V_i})^{(2)} =^{(sg)} u_i^{(2)} \circ \pi_2^{(0)}$$

- *Commutation update-update:*
  the order of storing in two different locations i and j does not matter. $\forall i \neq j \in Loc$,

$$u_j^{(2)} \circ (u_i \times id_{V_j})^{(2)} =^{(sg)} u_i^{(2)} \circ (id_{V_i} \times u_j)^{(2)}$$

Decorated proofs in [Dumas&Duval&Fousse&Reynaud 2011]

More decorated proofs (for exceptions)

- *Interaction catch-catch:*
  when catching an exception constructor i twice,
  the second catcher is never used. $\forall i \in \mathit{ExCstr}$,

$$\mathit{try}\ \{f\}\mathit{catch}\, i\, \{g\}\mathit{catch}\, i\, \{h\} =^{(sg)} \mathit{try}\ \{f\}\mathit{catch}\, i\, \{g\}$$

- *Commutation catch-catch:*
  when catching two different exception constructors i and j,
  the order of catching does not matter. $\forall i \neq j \in \mathit{ExCstr}$,

$$\mathit{try}\ \{f\}\mathit{catch}\, i\, \{g\}\mathit{catch}\, j\, \{h\} =^{(sg)} \mathit{try}\ \{f\}\mathit{catch}\, j\, \{h\}\mathit{catch}\, i\, \{g\}$$

Proof.

1. Start from the previous equations for states
2. Dualize
3. Encapsulate

Outline

Conclusion

- An effect is an apparent lack of soundness
- Designing proof systems from programming features: each computational effect has an associated logic
- States and exceptions may be considered as dual effects

Future work

- Using a proof assistant (Coq) for decorated proofs
- Combining effects by composing the spans of logics

A question

[Melliès 2010] About the notion of monad and
the notion of sheaf on a Grothendieck topology:

It is fascinating to observe that the most promising links between mathematics and programming languages emerged at these somewhat himalayan heights.

Mount Everest, 8 848 m.

Question. What is the “height” of our (naive?) approach?

Grand pic de Belledonne, 2 977 m.

Thanks for your attention