Diagrammatic specifications

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1 Introduction

This paper deals with some kind of progressive constructions of freely generated structures. For instance, in order to generate progressively the words on the alphabet $X = \{a, b\}$, we might first generate one word $ab$, then add it to $X$, getting $X_1 = \{a, b, ab\}$, and repeat a similar process from $X_1$. However, for this process to result in the construction of $X^*$, we must be able to remember that the string $ab$ in $X_1$ stands for the concatenation of $a$ and $b$. This means that we have to consider $X_1$ not just as a set, but as a partial monoid, with a partially defined concatenation operation which maps the pair $(a, b)$ towards the string $ab$. So, in this example, we have to deal with three different structures: the sets, the monoids, and the partial monoids. Clearly, the sets are the partial monoids where the operation is nowhere defined, and the monoids are the partial monoids where the operation is everywhere defined. Sets and monoids are used to define $X^*$ from $X$, while partial monoids are needed for building progressively $X^*$ from $X$.

Freely generated structures play a fundamental role in mathematics and computer science: for instance, words are freely generated from an alphabet, theorems from axioms, and programs from grammars. A freely generated structure may be quite “large”, however it happens that only a “small” part of it is needed: in order to prove a theorem from a given set of axioms, one does not prove all the theorems first…

It is well known that the categorical notion of adjunction is a basic one for dealing with freely generated structures [Mac Lane, 1971]. We are interested in adjunctions which allow some kind of progressive construction of the freely generated structure. For instance, the direct definition of $X^*$ from $X$ is an adjunction between sets and monoids, whereas the progressive construction of $X^*$ from $X$ needs the adjunction between partial monoids and monoids.

In this paper, we present a general framework for such adjunctions, which includes abstract definitions of syntactic entailment and semantic consequence. Our theory of diagrammatic specifications is derived, in a natural and simple way, from the theory of projective sketches. Thanks to the use of projective sketches at the metalevel, for the “specification of specifications”, the theory of diagrammatic specifications is quite homogeneous and effective.

Our motivation has its roots in the study of computer languages, mainly in the links between several programming styles, including imperative and object paradigms; these applications will be considered in subsequent papers.
Adjunctions, categories and projective sketches.

An adjunction is a pair of functors \((F : \mathcal{A}_1 \to \mathcal{A}_2, U : \mathcal{A}_2 \to \mathcal{A}_1)\) between two categories \(\mathcal{A}_1\) and \(\mathcal{A}_2\) such that, for all \(A_1\) in \(\mathcal{A}_1\) and \(A_2\) in \(\mathcal{A}_2\), the arrows from \(A_1\) to \(U(A_2)\) in \(\mathcal{A}_1\) are naturally in one-to-one correspondence with the arrows from \(F(A_1)\) to \(A_2\) in \(\mathcal{A}_2\):

\[
\text{Hom}_{\mathcal{A}_1}(F(A_1), A_2) \cong \text{Hom}_{\mathcal{A}_2}(A_1, U(A_2)).
\]

For instance, \(\mathcal{A}_1\) is the category of sets, \(\mathcal{A}_2\) is the category of monoids, the functor \(U\) maps each monoid to its underlying set, and the functor \(F\) maps each set to its freely generated monoid.

In addition, we introduce a meta-level, which is based on projective sketches. Sketches were introduced in [Ehresmann, 1966]. We assume that \(\mathcal{A}_1\) is the category of realizations (i.e. models) of a projective sketch \(\mathcal{E}_1\), and that \(\mathcal{A}_2\) is the category of realizations of a projective sketch \(\mathcal{E}_2\). Such categories are known as locally presentable categories [Gabriel and Ulmer, 1971]. We also assume that \(U\) and \(F\) are the underlying functor and the freely generating functor associated to a propagator (i.e. an homomorphism) \(P : \mathcal{E}_1 \to \mathcal{E}_2\):

\[
\mathcal{A}_1 = \text{Real}(\mathcal{E}_1), \quad \mathcal{A}_2 = \text{Real}(\mathcal{E}_2), \quad U = U_P : \text{Real}(\mathcal{E}_2) \to \text{Real}(\mathcal{E}_1), \quad F = F_P : \text{Real}(\mathcal{E}_1) \to \text{Real}(\mathcal{E}_2).
\]

For instance, \(\mathcal{E}_1\) is a projective sketch of sets, which means that the realizations of \(\mathcal{E}_1\) are the sets and their morphisms are the maps, and \(\mathcal{E}_2\) is a projective sketch of monoids, which means that the realizations of \(\mathcal{E}_2\) are the monoids and their morphisms are the morphisms of monoids.

Decomposition.

In this context, the adjunctions which allow a progressive construction of the freely generating functor \(F\) are those with a full and faithful underlying functor \(U\); then, we say that the propagator is fractioning. On the other hand, we say that a propagator is filling whenever the freely generating functor \(F\) is full and faithful. Both words “fractioning” and “filling” stem from properties of these propagators which are stated in the paper.

We prove that any propagator \(P : \mathcal{E}_0 \to \mathcal{F}\) can be decomposed as \(P = K \circ J\), with \(J : \mathcal{E}_0 \to \mathcal{E}\) filling and \(K : \mathcal{E} \to \mathcal{F}\) fractioning. In addition, \(J\) can be chosen in such a way that the construction of \(F_J\) is trivial. A consequence of the decomposition \(P = K \circ J\) is that \(F_P(A_0) = F_K(F_J(A_0))\) for all realization \(A_0\) of \(\mathcal{E}_0\). So, in order to build \(F_P(A_0)\), we may replace the propagator \(P\) and the realization \(A_0\) of \(\mathcal{E}_0\) by the fractioning propagator \(K\) and the realization \(F_J(A_0)\) of \(\mathcal{E}\).

For instance, when \(P\) is a propagator from a sketch \(\mathcal{E}_0\) of sets to a sketch \(\mathcal{F}\) of monoids, then \(\mathcal{E}\) can be a sketch of partial monoids. The partial monoid \(F_J(X)\) is still denoted \(X\): it is the set \(X\) with the nowhere-defined partial concatenation.

Specifications.

Let \(P : \mathcal{E}_0 \to \mathcal{F}\) be a propagator together with a decomposition \(P = K \circ J\), with \(J : \mathcal{E}_0 \to \mathcal{E}\) filling and \(K : \mathcal{E} \to \mathcal{F}\) fractioning, as above. In this context, let us give some basic definitions, followed by an example.

The specifications are the realizations of \(\mathcal{E}\), the signatures are the realizations of \(\mathcal{E}_0\), and the domains are the realizations of \(\mathcal{F}\). Then, the set of models of a specification \(S\) with values in a domain \(D\) is:

\[
\text{Mod}_K(S, D) = \text{Hom}_{\text{Real}(\mathcal{F})}(F_K(S), D),
\]

so that, by adjunction:

\[
\text{Mod}_K(S, D) \cong \text{Hom}_{\text{Real}(\mathcal{E})}(S, U_K(D)).
\]

For example, let us look at simple equational specifications, where all the operators are unary. A composite graph is a directed graph together with a partial composition of arrows, so that a category is a composite graph with total composition of arrows. Then, a signature is a composite graph, a specification is a signature together with a binary relation \(\equiv\) on arrows \((f \equiv g\) is called an equation), and a
domain is a category together with a binary relation \( \equiv \) on arrows which is a congruence, i.e., an equivalence relation compatible with the composition. So, \( S_0 \) is a projective sketch of compositio graphs, \( E \) is a projective sketch of compositio graphs with equations, and \( \overline{E} \) is a projective sketch of categories with congruence. The propagators \( P, J \) and \( K \) are straightforward.

In order to specify the integers, we consider the signature \( S_{\text{int},0} \) which is made of a point \( I \), four arrows \( s, p, s \circ p \) and \( p \circ s : I \to I \), and the partial composition which maps the pair \((p, s)\) to the arrow \( s \circ p \) and the pair \((s, p)\) to the arrow \( p \circ s \). The signature \( S_{\text{int},0} \) together with the equation \( p \circ s \equiv s \circ p \) is a specification \( S_{\text{int}} \). The signature \( U_j(S_{\text{int}}) \) which is underlying \( S_{\text{int}} \) is \( S_{\text{int},0} \). The domain \( F_K(S_{\text{int}}) \) which is freely generated by \( S_{\text{int}} \) is the category with one point \( I \) and all arrows composed from \( s \) and \( p \) (like \( s \circ s \circ p \circ p \circ s \circ p \)), and with the congruence relation \( f \equiv g \) if and only if the number of \( s \)'s minus the number of \( p \)'s is the same one in \( f \) and in \( g \). The arrows of \( F_K(S_{\text{int}}) \) are the terms and its equations are the theorems which are derived from the specification \( S_{\text{int}} \).

On the other hand, let \( D_{\text{set}} \) be the realization of \( \overline{E} \) with the sets as points, the maps as arrows, and the equality as congruence. A set-valued model of \( S_{\text{int}} \) can be seen either as a morphism from \( F_K(S_{\text{int}}) \) to \( D_{\text{set}} \) in Real(\( \overline{E} \)), or as a morphism from \( S_{\text{int}} \) to \( U_K(D_{\text{set}}) \) in Real(\( E \)): it interprets each point of \( S_{\text{int}} \) as a set, each arrow of \( S_{\text{int}} \) as a map, and each equation of \( S_{\text{int}} \) as an equality. For instance, there is a set-valued model of the specification \( S_{\text{int}} \) which maps the point \( I \) on the set of integers, the arrows \( s \) and \( p \) on the successor and predecessor map respectively, and the arrows \( p \circ s \) and \( s \circ p \) on the identity.

Diagrammatic specifications are not restricted to equational ones. Actually, first-order and higher-order specifications also can be considered as diagrammatic specifications.

Entailment and consequence.

Now let us focus on a fractioning propagator \( K : E \to \overline{E} \).

A morphism of specifications \( \sigma : S \to S' \) is a syntactic entailment, which is denoted \( S \frac{\sigma}{\rightarrow} S' \), whenever the derived morphism \( F_K(\sigma) : F_K(S) \to F_K(S') \) is an isomorphism:

\[
S \frac{\sigma}{\rightarrow} S' \quad \text{if and only if} \quad F_K(S) \xrightarrow{\cong} F_K(S').
\]

In addition, a syntactic entailment can be obtained by a succession of deduction steps, using the deduction rules which are given by the projective sketch \( \overline{E} \).

A morphism of specifications \( \sigma : S \to S' \) is a semantic consequence with respect to some domain \( D \), which is denoted \( S \frac{\sigma}{\rightarrow_D} S' \), whenever the derived morphism \( \text{Mod}_K(\sigma, D) : \text{Mod}_K(S', D) \to \text{Mod}_K(S, D) \) is an isomorphism:

\[
S \frac{\sigma}{\rightarrow_D} S' \quad \text{if and only if} \quad \text{Mod}_K(S, D) \cong \text{Mod}_K(S', D).
\]

In addition, a semantic consequence can also be defined from a satisfaction relation between models and specifications, which makes sense only when a filling propagator \( J : E_0 \to E \) is given, besides the fractioning propagator \( K : E \to \overline{E} \).

With these definitions, the soundness property is satisfied, which means that syntactic entailment implies semantic consequence:

\[
\text{if} \quad S \frac{\sigma}{\rightarrow} S' \quad \text{then} \quad S \frac{\sigma}{\rightarrow_D} S' \quad \text{for all} \ D.
\]

Our diagrammatic specifications can be related to the institutions [Goguen and Burstall, 1992]. Then, it is possible to compare our notions of entailment and consequence with the notions which occur in logic with the same names and the symbols \( \vdash \) and \( \models \), respectively. The fractioning propagator \( K \) can be chosen in such a way that there is a \( K \)-domain "of sets" \( D_{\text{set}} \) and a \( K \)-specification \( S \) corresponds to a conjunction of sentences \( \varphi_1, \varphi_2, \ldots, \varphi_k \). Then a morphism \( \sigma : S \to S' \) can correspond to adding a sentence \( \psi \), so that the \( K \)-specification \( S' \) corresponds to the conjunction of the sentences \( \varphi_1, \varphi_2, \ldots, \varphi_k, \psi \). In such a situation:

\[
S \frac{\sigma}{\rightarrow} S' \quad \text{if and only if} \quad \varphi_1, \varphi_2, \ldots, \varphi_k \vdash \psi ;
\]

\[
S \frac{\sigma}{\rightarrow_D} S' \quad \text{if and only if} \quad \varphi_1, \varphi_2, \ldots, \varphi_k \models \psi .
\]
Organization of the paper.

This paper begins with a review of some useful definitions and results about categories (section 2). These results are well known, they can be found in [Mac Lane, 1971] for example.

Then in section 3 are reviewed some definitions and results about projective sketches, which are not so well known, although most of them can be found in [Coppey and Lair, 1984] and [Coppey and Lair, 1988], or in [Duval and Lair, 2001].

Section 4 is devoted to the study of fractioning and filling propagators and to the decomposition theorem.

In section 5 are defined the notions of specification, domain and model, as well as syntactic entailment and semantic consequence.

Finally, in section 6, we look at equational diagrammatic specifications and we outline some links between diagrammatic specifications and institutions.

The applications of diagrammatic specifications to the study of computer languages will be the subject of forthcoming papers.

From the point of view of terminology, we have made some choices: point rather than object, source and target rather than domain and codomain, and so on. For technical issues, including the size issues, we refer to the reference manual [Duval and Lair, 2001]. So, for instance, we speak without any care about the category of categories.

Moreover, in order to keep distinct the specification level and the meta-specification level, we speak on one side about morphisms and models of specifications, and on the other side about propagators and realizations of projective sketches.

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2 Categories, adjoints, limits

Here are a few basic facts about categories, adjunction and limits. All this is very well known, it goes back to Eilenberg and Mac Lane in the 1940's, and can be found in [Mac Lane, 1971] for instance. However, the flavour of our definition of limits stems from the theory of sketches. In addition, some of our illustrations will be given a precise status in section 4.

2.1 Directed graphs

Definition 2.1.1

A (directed) graph $\mathcal{G}$ is made of a set of points, a set of arrows, and two maps from arrows to points, which assign to each arrow respectively its source and its target.

An arrow $g$ with source $G_1$ and target $G_2$, i.e. an arrow from $G_1$ to $G_2$, is denoted $g : G_1 \to G_2$ or $G_1 \xrightarrow{g} G_2$. The set of arrows from $G_1$ to $G_2$ in $\mathcal{G}$ is denoted $\text{Hom}_\mathcal{G}(G_1, G_2)$.

An arrow $g$ is a loop if $G \xrightarrow{g} G$.

Two arrows $g_1$ and $g_2$ are consecutive if $G_1 \xrightarrow{g_1} G_2 \xrightarrow{g_2} G_3$.

A triple of arrows $(g_1, g_2, g)$ is a triangle if $G_1 \xrightarrow{g_1} G_2 \xrightarrow{g_2} G_3$ and $G_1 \xrightarrow{g} G_3$.

The opposite of $\mathcal{G}$ is the directed graph $\mathcal{G}^{op}$ with the arrows in the opposite direction.

Definition 2.1.2

A graph homomorphism $H : \mathcal{G} \to \mathcal{G}'$ is made of two maps, both denoted $H$, from the points (resp. the arrows) of $\mathcal{G}$ towards the points (resp. the arrows) of $\mathcal{G}'$, such that if $g : G_1 \to G_2$ then $H(g) : H(G_1) \to H(G_2)$.
Hence, for all points $G_1$ and $G_2$ in $\mathcal{G}$, the map $H$ on arrows restricts to a map:

$$H_{G_1, G_2} : \text{Hom}_G(G_1, G_2) \to \text{Hom}_{G'}(H(G_1), H(G_2)).$$

An inclusion $\mathcal{G} \subseteq \mathcal{G}'$ is a graph homomorphism $H : \mathcal{G} \to \mathcal{G}'$ which is an inclusion both on the sets of points and on the sets of arrows.

A contravariant graph homomorphism $H : \mathcal{G} \to \mathcal{G}'$ is made of two maps, both denoted $H$, from the points (resp. the arrows) of $\mathcal{G}$ towards the points (resp. the arrows) of $\mathcal{G}'$, such that if $g : G_1 \to G_2$ then $H(g) : H(G_2) \to H(G_1)$.

A contravariant graph homomorphism $H : \mathcal{G} \leftrightarrow \mathcal{G}'$ can be identified either to a graph homomorphism $\mathcal{G}^{op} \to \mathcal{G}'$ or to a graph homomorphism $\mathcal{G} \to (\mathcal{G}')^{op}$.

A graph can be illustrated, as usual. For instance, here is an illustration of the graph made of a triangle:

\begin{center}
\begin{tikzpicture}
    \node (A) at (0,0) {$G_1$};
    \node (B) at (2,2) {$G_2$};
    \node (C) at (2,-2) {$G_3$};
    \path[->]
    (A) edge node [above] {$g_1$} (B)
    (A) edge node [below] {$g$} (C)
    (B) edge node [above] {$g_2$} (C);
\end{tikzpicture}
\end{center}

### 2.2 Categories

**Definition 2.2.1**

A category $\mathcal{A}$ is made of a directed graph $\text{Supp}(\mathcal{A})$, called the support of $\mathcal{A}$, together with:

- for each point $A$, a loop $\text{id}_A : A \to A$ which is called the identity at $A$,
- for each consecutive pair of arrows $(a_1, a_2)$, a triangle $(a_1, a_2, a_2 \circ a_1)$ where $a_2 \circ a_1$ is called the composite of $a_1$ and $a_2$,

which satisfies the following unitarity and associativity properties:

- $a \circ \text{id}_A = a$ and $\text{id}_A \circ a = a$ for all arrow $A_1 \xrightarrow{a} A_2$,
- $(a_3 \circ a_2) \circ a_1 = a_3 \circ (a_2 \circ a_1)$ (which is denoted $a_3 \circ a_2 \circ a_1$) for all triple of consecutive arrows $A_1 \xrightarrow{a_1} A_2 \xrightarrow{a_2} A_3 \xrightarrow{a_3} A_4$.

The following definitions hold in any category $\mathcal{A}$.

An isomorphism $a : A_1 \xrightarrow{a} A_2$ is an arrow which has an inverse: there is an arrow $a' : A_2 \to A_1$ such that $a' \circ a = \text{id}_{A_1}$ and $a \circ a' = \text{id}_{A_2}$.

A monomorphism is an arrow $a : A_1 \to A_2$ such that, for all $A$ and all $a', a'' : A \to A_1$, if $a \circ a' = a \circ a''$ then $a' = a''$.

A split monomorphism is an arrow $a : A_1 \to A_2$ together with a left inverse, i.e. with an arrow $a' : A_2 \to A_1$ such that $a' \circ a = \text{id}_{A_1}$; then $a$ is a monomorphism.

An epimorphism is an arrow $a : A_1 \to A_2$ such that, for all $A$ and all $a', a'' : A_2 \to A$, if $a' \circ a = a'' \circ a$ then $a' = a''$.

A split epimorphism is an arrow $a : A_1 \to A_2$ together with a right inverse, i.e. with an arrow $a' : A_2 \to A_1$ such that $a \circ a' = \text{id}_{A_2}$; then $a$ is an epimorphism.

**Definition 2.2.2**

Let $\mathcal{A}$ and $\mathcal{A}'$ be two categories. A functor $H : \mathcal{A} \to \mathcal{A}'$ is a graph homomorphism $\text{Supp}(H) : \text{Supp}(\mathcal{A}) \to \text{Supp}(\mathcal{A}')$ which preserves identities and composites.

An inclusion $\mathcal{A} \subseteq \mathcal{A}'$ is a functor such that its support is an inclusion of graphs; then, $\mathcal{A}$ is a subcategory of $\mathcal{A}'$.

A contravariant functor $H : \mathcal{A} \to \mathcal{A}'$ is a contravariant graph homomorphism $\text{Supp}(H) : \text{Supp}(\mathcal{A}) \to \text{Supp}(\mathcal{A}')$ which preserves identities and composites.
Let $\mathcal{A}$ be a category.
For all points $A$ of $\mathcal{A}$, the functor $\text{Hom}_\mathcal{A}(A,-) : \mathcal{A} \to \text{Set}$ maps each point $B$ of $\mathcal{A}$ to the set $\text{Hom}_\mathcal{A}(A,B)$ and each arrow $b : B_1 \to B_2$ of $\mathcal{A}$ to the map $\text{Hom}_\mathcal{A}(A,b) : \text{Hom}_\mathcal{A}(A,B_1) \to \text{Hom}_\mathcal{A}(A,B_2)$ such that $\text{Hom}_\mathcal{A}(A,b)(c_1) = b \circ c_1$.
For all points $B$ of $\mathcal{A}$, the contravariant functor $\text{Hom}_\mathcal{A}(-,B) : \mathcal{A} \to \text{Set}$ maps each point $A$ of $\mathcal{A}$ to the set $\text{Hom}_\mathcal{A}(A,B)$ and each arrow $a : A_1 \to A_2$ of $\mathcal{A}$ to the map $\text{Hom}_\mathcal{A}(a,B) : \text{Hom}_\mathcal{A}(A_2,B) \to \text{Hom}_\mathcal{A}(A_1,B)$ such that $\text{Hom}_\mathcal{A}(a,B)(c_2) = c_2 \circ a$.

\[ \begin{array}{ccc}
A & \xrightarrow{c_1} & B_1 \\
\downarrow{b} & & \downarrow{b} \\
B_2 & \xrightarrow{c_2} & B_1 \\
\end{array} \]

**Definition 2.2.3**

Let $H : \mathcal{A} \to \mathcal{A}'$ be a functor between two categories. For all points $A_1$ and $A_2$ in $\mathcal{A}$, there is a map $H_{A_1,A_2} : \text{Hom}_\mathcal{A}(A_1,A_2) \to \text{Hom}_{\mathcal{A}'}(H(A_1),H(A_2))$.
- The functor $H$ is faithful if for all points $A_1$ and $A_2$ in $\mathcal{A}$, the map $H_{A_1,A_2}$ is injective.
- The functor $H$ is full if for all points $A_1$ and $A_2$ in $\mathcal{A}$, the map $H_{A_1,A_2}$ is surjective.

If a functor $H : \mathcal{A} \to \mathcal{A}'$ is an inclusion and is full, then $\mathcal{A}$ is a full subcategory of $\mathcal{A}'$.

**Definition 2.2.4**

Let $\mathcal{A}$ and $\mathcal{A}'$ be two categories and $H_1,H_2 : \mathcal{A} \to \mathcal{A}'$ two functors. A natural transformation $\tau : H_1 \Rightarrow H_2 : \mathcal{A} \to \mathcal{A}'$ is made of an arrow $\tau_A : H_1(A) \to H_2(A)$ of $\mathcal{A}'$ for each point $A$ of $\mathcal{A}$, such that $\tau_{A_2} \circ H_1(a) = H_2(a) \circ \tau_{A_1}$ in $\mathcal{A}'$ for each arrow $a : A_1 \to A_2$ of $\mathcal{A}$.

\[ \begin{array}{ccc}
H_1(A_1) & \xrightarrow{\tau_{A_1}} & H_2(A_1) \\
H_1(a) & \downarrow & H_2(a) \\
H_1(A_2) & \xrightarrow{\tau_{A_2}} & H_2(A_2) \\
\end{array} \]

This can be expressed as follows: for all $A$ in $\mathcal{A}$ there is an arrow $\tau_A : H_1(A) \to H_2(A)$ of $\mathcal{A}'$ which is natural in $A$.

A natural isomorphism is a natural transformation $\tau : H_1 \Rightarrow H_2$ such that the arrow $\tau_A$ is an isomorphism in $\mathcal{A}'$ for all point $A$ of $\mathcal{A}$.

For all functor $H : \mathcal{A} \to \mathcal{A}'$, the identity $\text{id}_H : H \Rightarrow H$ is the natural transformation such that $(\text{id}_H)_A = \text{id}_{H(A)}$ for all point $A$ of $\mathcal{A}$.

For all pair of consecutive natural transformations $H_1 \xRightarrow{\nu_1} H_2 \xRightarrow{\nu_2} H_3$, where $H_1,H_2,H_3 : \mathcal{A} \to \mathcal{A}'$, the composite $H_1 \xRightarrow{\nu_3} H_3$ is the natural transformation such that $(\nu_3)_A = (\nu_2)_A \circ (\nu_1)_A$ for all point $A$ of $\mathcal{A}$.

For all functor $H : \mathcal{A} \to \mathcal{A}'$ and all natural transformation $\tau' : H'_1 \Rightarrow H'_2 : \mathcal{A}' \to \mathcal{A''}$, the composite $\tau' \circ H : H'_1 \circ H \Rightarrow H'_2 \circ H : \mathcal{A} \to \mathcal{A''}$ is the natural transformation such that $(\tau' \circ H)_A = \tau'_{H(A)} : H'_1(H(A)) \to H'_2(H(A))$ for all point $A$ of $\mathcal{A}$.

For all natural transformation $\tau : H_1 \Rightarrow H_2 : \mathcal{A} \to \mathcal{A}'$ and all functor $H' : \mathcal{A}' \to \mathcal{A''}$, the composite $H' \circ \tau : H'_1 \circ H_1 \Rightarrow H'_2 \circ H_2 : \mathcal{A} \to \mathcal{A''}$ is the natural transformation such that $(H' \circ \tau)_A = H'(\tau_A) : H'(H_1(A)) \to H'(H_2(A))$ for all point $A$ of $\mathcal{A}$.

**Example 2.2.5**

Until now, up to some care about size issues, we may define the following categories:
- $\text{Set}$ is the category of sets and maps. In this category, an isomorphism is a bijection, a monomorphism is an injection, and an epimorphism is a surjection. A split monomorphism is an injection together with a chosen retraction and a split epimorphism is a surjection together with a chosen section.
- $\mathcal{Gr}$ is the category of directed graphs and graph homomorphisms.
- $\mathbf{Cat}$ is the category of categories and functors.
- For all categories $\mathcal{A}$ to $\mathcal{A}'$, $\mathbf{Func}(\mathcal{A}, \mathcal{A}')$ is the category of functors from $\mathcal{A}$ to $\mathcal{A}'$ and natural transformations. It is easy to check that a natural isomorphism is an isomorphism of the category $\mathbf{Func}(\mathcal{A}, \mathcal{A}')$.

There are several functors between these categories.
- There is a functor $\mathbf{Set} \to \mathcal{G}r$, which maps a set $X$ to the graph with $X$ as its set of points and with no arrow; this functor identifies $\mathbf{Set}$ with a full subcategory of $\mathcal{G}r$.
- The functor $\mathbf{Pt} : \mathcal{G}r \to \mathbf{Set}$ maps each graph to its set of points and each graph homomorphism to the underlying map on points.
- The functor $\mathbf{Ar} : \mathcal{G}r \to \mathbf{Set}$ maps each graph to its set of arrows and each graph homomorphism to the underlying map on arrows.
- The functor $\mathbf{Supp} : \mathbf{Cat} \to \mathcal{G}r$ maps each category to its underlying graph and each functor to its underlying graph homomorphism.

### 2.3 Adjunction

**Definition 2.3.1**

Let $\mathcal{A}$ and $\mathcal{A}'$ be categories. An adjunction from $\mathcal{A}$ to $\mathcal{A}'$ is a pair of functors:

$$(\mathcal{A} \xrightarrow{F} \mathcal{A}', \mathcal{A} \xleftarrow{U} \mathcal{A}')$$

together with, for all points $A$ of $\mathcal{A}$ and $A'$ of $\mathcal{A}'$, a bijection which is natural in $A$ and $A'$:

$$\text{Hom}_\mathcal{A}(A, U(A')) \cong \text{Hom}_{\mathcal{A}'}(F(A), A').$$

Then, $F$ is a left adjoint for $U$, and $U$ is a right adjoint for $F$. If a functor $U$ has a left adjoint, it is unique up to a natural isomorphism. If a functor $F$ has a right adjoint, it is unique up to a natural isomorphism.

**Theorem 2.3.2 (adjunction)**

An adjunction $(F, U)$ from $\mathcal{A}$ to $\mathcal{A}'$ determines two natural transformations:

$$\eta : \text{id}_\mathcal{A} \Rightarrow U \circ F : \mathcal{A} \to \mathcal{A} \quad \text{and} \quad \varepsilon : F \circ U \Rightarrow \text{id}_{\mathcal{A}'} : \mathcal{A}' \to \mathcal{A}'$$

such that for all points $A$ of $\mathcal{A}$ and $A'$ of $\mathcal{A}'$, the bijection $\text{Hom}_\mathcal{A}(A, U(A')) \cong \text{Hom}_{\mathcal{A}'}(F(A), A')$ maps $a : A \to U(A')$ towards:

$$a^* = \varepsilon_{A'} \circ F(a) : F(A) \to A',$$

and maps $a' : F(A) \to A'$ towards:

$$a'_* = U(a') \circ \eta_A : A \to U(A').$$

In addition, both composite natural transformations below are identities:

$$U \xrightarrow{\eta U} U \circ F \circ U \xrightarrow{U \varepsilon} U \quad \text{and} \quad F \xrightarrow{\varepsilon F} F \circ U \circ F \xrightarrow{F \eta} F.$$

This result is proven in [Mac Lane, 1971, p. 80]. The last assertion means that for all points $A$ of $\mathcal{A}$ and $A'$ of $\mathcal{A}'$, $U(\varepsilon_{A'}) \circ \eta_{U(A)} = \text{id}_{U(A)}$ and $\varepsilon_{F(A)} \circ F(\eta_A) = \text{id}_{F(A)}$.

It follows from this theorem that $\varepsilon_{A'} = (\text{id}_{U(A')})^*$ and $\eta_A = (\text{id}_{F(A)})_*$.

**Definition 2.3.3**

Let $(F, U)$ be an adjunction from $\mathcal{A}$ to $\mathcal{A}'$.
- The natural transformation $\eta : \text{id}_\mathcal{A} \Rightarrow U \circ F : \mathcal{A} \to \mathcal{A}$ is the unit of the adjunction.
- The natural transformation $\varepsilon : F \circ U \Rightarrow \text{id}_{\mathcal{A}'} : \mathcal{A'} \to \mathcal{A}'$ is the counit of the adjunction.
- The functor $M = U \circ F : \mathcal{A} \to \mathcal{A}$ together with the natural transformations $\eta : \text{id}_{\mathcal{A}} \Rightarrow M : \mathcal{A} \to \mathcal{A}$ and $\mu = U \circ \varepsilon \circ F : M^2 \Rightarrow M : \mathcal{A} \to \mathcal{A}$ is the monad associated to the adjunction. The functor $M$ is the endofunctor of the monad, while $\eta$ is its unit and $\mu$ is its multiplication; it is made of the maps $\mu_A = U(\varepsilon_{F(A)^{\mathcal{A}}}) : U(F(U(F(A)))) \Rightarrow U(F(A))$.
- A monad $(M, \eta, \mu)$ is idempotent when $\mu$ is a natural isomorphism.

A monad $(M, \eta, \mu)$ gives rise to natural transformations $M \circ \eta : M \Rightarrow M^2$ and $\eta \circ M : M \Rightarrow M^2$. Generally, these natural transformations are distinct, however each of them is a right inverse for the multiplication: this is known as the unitarity property of the monad $M$:

$$\mu \circ (M \circ \eta) = \text{id}_M \text{ and } \mu \circ (\eta \circ M) = \text{id}_M.$$ 

This means that $\mu_A \circ M(\eta_A) = \text{id}_A$ and $\mu_A \circ \eta_{M(A)} = \text{id}_A$ for all point $A$ of $\mathcal{A}$.

There is also an associativity property of the monad $M$, which we will not use.

Now, we focus on adjunctions $(F, U)$ where either $U$ or $F$, or both, is full and faithful. Let $(F, U)$ be an adjunction from $\mathcal{A}$ to $\mathcal{A}'$, with unit $\eta : \text{id}_{\mathcal{A}} \Rightarrow U \circ F$ and counit $\varepsilon : F \circ U \Rightarrow \text{id}_{\mathcal{A}'}$.

**Theorem 2.3.4 (full and faithful functors in adjunctions)**
- The functor $U$ is full and faithful if and only if $\varepsilon$ is a natural isomorphism.
- The functor $F$ is full and faithful if and only if $\eta$ is a natural isomorphism.

The first part of this theorem is proven in [Mac Lane, 1971, p. 88], the second part can be proven in a dual way.

Let $A$ be a point of $\mathcal{A}$ and $A'$ a point of $\mathcal{A}'$. Theorem 2.3.4 proves that:
- if $U$ is full and faithful and if it is an inclusion $\mathcal{A}' \subseteq \mathcal{A}$, then $F(A') \cong A'$ as soon as $A'$ is in $\mathcal{A}'$.
- if $F$ is full and faithful and if it is an inclusion $\mathcal{A} \subseteq \mathcal{A}'$, then $U(A) \cong A$ as soon as $A$ is in $\mathcal{A}$.

**Corollary 2.3.5 (full and faithful $U$ or $F$)**
If either $U$ or $F$ is full and faithful, then the following natural transformations are natural isomorphisms:
- $\eta \circ U : U \cong U \circ F \circ U$, with inverse $U \circ \varepsilon$,
- $\varepsilon \circ F : F \circ U \circ F \cong F$, with inverse $F \circ \eta$,
- $\mu : M^2 \cong M$, with inverse $\eta \circ M = M \circ \eta$: the monad $(M, \eta, \mu)$ is idempotent.

**Definition 2.3.6**
An equivalence $\mathcal{A} \simeq \mathcal{A}'$ between two categories $\mathcal{A}$ and $\mathcal{A}'$ is a pair of functors $(H : \mathcal{A} \to \mathcal{A}', H' : \mathcal{A}' \to \mathcal{A})$ and a pair of natural isomorphisms $H' \circ H \cong \text{id}_{\mathcal{A}}$ and $H \circ H' \cong \text{id}_{\mathcal{A}'}$.

This definition can be considered as a weakened notion of isomorphism. Indeed, according to the general definition of an isomorphism in a category, applied to the category $\text{Cat}$, an isomorphism $\mathcal{A} \cong \mathcal{A}'$ between two categories $\mathcal{A}$ and $\mathcal{A}'$ is a pair of functors $(H : \mathcal{A} \to \mathcal{A}', H' : \mathcal{A}' \to \mathcal{A})$ such that $H' \circ H = \text{id}_{\mathcal{A}}$ and $H \circ H' = \text{id}_{\mathcal{A}'}$.

So, theorem 2.3.4 states that when both functors $U$ and $F$ in an adjunction $(F, U)$ are full and faithful, they determine an equivalence between the categories $\mathcal{A}$ and $\mathcal{A}'$. It can be proven that in this way we get all the equivalences of categories [Mac Lane, 1971, p.91].

**2.4 Yoneda lemma**

Let $\mathcal{A}$ be a category. Then $\text{Func}(\mathcal{A}, \text{Set})$ is the category of functors from $\mathcal{A}$ to $\text{Set}$, with the natural transformations as arrows. For all points $A$ and $B$ of $\mathcal{A}$, the functors $\text{Hom}_{\mathcal{A}}(A, -) : \mathcal{A} \to \text{Set}$ and $\text{Hom}_{\mathcal{A}}(-, B) : \mathcal{A} \to \text{Set}$ are defined in section 2.2.
Definition 2.4.1
The Yoneda contravariant functor associated to $\mathcal{A}$:

$$Y_{\mathcal{A}} : \mathcal{A} \rightarrow \text{Func}(\mathcal{A}, \text{Set})$$

is such that:
- $Y_{\mathcal{A}}(A) = \text{Hom}_{\mathcal{A}}(A, -) : \mathcal{A} \rightarrow \text{Set}$ for all point $A$ of $\mathcal{A}$,
- $Y_{\mathcal{A}}(a) = \text{Hom}_{\mathcal{A}}(a, -) : Y_{\mathcal{A}}(A_2) \rightarrow Y_{\mathcal{A}}(A_1) : \mathcal{A} \rightarrow \text{Set}$ for all arrow $a : A_1 \rightarrow A_2$ of $\mathcal{A}$.

Then, for all point $A$ of $\mathcal{A}$, the set $(Y_{\mathcal{A}}(A))(A) = \text{Hom}_{\mathcal{A}}(A, A)$ contains $id_A$. So, for all functor $H : \mathcal{A} \rightarrow \text{Set}$, there is a map $\text{Hom}_{\text{Func}(\mathcal{A}, \text{Set})}(Y_{\mathcal{A}}(A), H) \rightarrow H(A)$ which maps each natural transformation $\tau : Y_{\mathcal{A}}(A) \Rightarrow H$ to the element $\tau_A(id_A) \in H(A)$.

**Theorem 2.4.2 (Yoneda lemma)** The Yoneda contravariant functor $Y_{\mathcal{A}} : \mathcal{A} \rightarrow \text{Func}(\mathcal{A}, \text{Set})$ is full and faithful. In addition, for each point $A$ of $\mathcal{A}$ and each functor $H$ from $\mathcal{A}$ to $\text{Set}$, naturally in $A$ and in $H$, the map $\tau \mapsto \tau_A(id_A)$ is a bijection:

$$\text{Hom}_{\text{Func}(\mathcal{A}, \text{Set})}(Y_{\mathcal{A}}(A), H) \cong H(A).$$

Let $\mathbb{I}$ denote a one-element set. Then $X = \text{Hom}_{\text{Set}}(\mathbb{I}, X)$ for each set $X$, so that the bijection in the Yoneda lemma can be stated as the property of a freely generated structure:

$$\text{Hom}_{\text{Set}}(\mathbb{I}, ev_A(H)) \cong \text{Hom}_{\text{Func}(\mathcal{A}, \text{Set})}(Y_{\mathcal{A}}(A), H),$$

naturally in $H$. So, $Y_{\mathcal{A}}(A)$ is free over $\mathbb{I}$, with respect to the functor $ev_A$ [Ehresmann, 1965].

**Example 2.4.3**
Let us look at some functors from the examples in section 2.1.
- The functor $Pt : \mathcal{G} \rightarrow \text{Set}$ has a left adjoint, which is the inclusion functor $\text{Set} \subseteq \mathcal{G}$.
- The functor $U_{\mathcal{G}}, \text{cat} = \text{Supp} : \text{Cat} \rightarrow \mathcal{G}$ has a left adjoint $F_{\mathcal{G}}, \text{cat} : \mathcal{G} \rightarrow \text{Cat}$, which maps a graph $\mathcal{G}$ to the category $F_{\mathcal{G}}, \text{cat}(\mathcal{G})$ with the same points as $\mathcal{G}$, and with arrows the paths of $\mathcal{G}$, which are obtained by composing any number of consecutive arrows of $\mathcal{G}$ (considering that the identity arrows of $F_{\mathcal{G}}, \text{cat}(\mathcal{G})$ are composed of no arrow of $\mathcal{G}$).
- The functor $Pt : \mathcal{G} \rightarrow \text{Set}$ is neither full nor faithful; its left adjoint $\text{Set} \subseteq \mathcal{G}$ is full and faithful.
- The functor $U_{\mathcal{G}}, \text{cat}$ and its left adjoint $F_{\mathcal{G}}, \text{cat}$ are both faithful, but none is full.

2.5 Compositional graphs

In order to define limits in a category, in section 2.6, we will use graphs with “some” identities and composites.

**Definition 2.5.1**
A compositional graph $\mathcal{G}$ is made of a directed graph $\text{Supp}(\mathcal{G})$, called the support of $\mathcal{G}$, together with:
- for some points $A$, a loop $A \xrightarrow{id_A} A$ which is called the identity at $A$,
- for some consecutive pairs of arrows $(a_1, a_2)$, a triangle $(a_1, a_2, a_2 \circ a_1)$ where $a_2 \circ a_1$ is called the composite of $a_1$ and $a_2$.

The unitarity and associativity properties do not hold in a compositional graph.

Let $\mathcal{G}$ and $\mathcal{G}'$ be two compositional graphs. A functor $H : \mathcal{G} \rightarrow \mathcal{G}'$ is a graph homomorphism $\text{Supp}(H) : \text{Supp}(\mathcal{G}) \rightarrow \text{Supp}(\mathcal{G}')$ which preserves identities and composites.

An inclusion $\mathcal{G} \subseteq \mathcal{G}'$ is a functor such that its support is an inclusion of graphs.

A contravariant functor $H : \mathcal{G} \rightarrow \mathcal{G}'$ is a contravariant graph homomorphism $\text{Supp}(H) : \text{Supp}(\mathcal{G}) \rightarrow \text{Supp}(\mathcal{G}')$ which preserves identities and composites.
A composite graph can be illustrated as its support together with the notations \( id_G \) for identities, and \( g_2 \circ g_1 \) or \( \cup \) for composites. Some identities and composites may be omitted. For instance, here are three illustrations of a composite graph made of a commutative triangle:

\[
\begin{array}{ccc}
G_1 & \xrightarrow{g_1} & G_2 \\
\downarrow g & & \downarrow g_2 \\
G_3 & \xrightarrow{g=g_2 \circ g_1} & G_2
\end{array}
\]

\[
\begin{array}{ccc}
G_1 & \xrightarrow{\cup} & G_2 \\
\downarrow g & & \downarrow g_2 \\
G_3 & \xrightarrow{g=g_2 \circ g_1} & G_2
\end{array}
\]

A category \( \mathcal{A} \) can be identified to a composite graph where there is an identity at each point, a composite for each consecutive pair of arrows, and which satisfies the unitarity and associativity properties. Then, a functor between categories is a functor of composite graphs.

The definition of natural transformations between functors from a composite graph to a category is an easy generalization of the definition of natural transformations in section 2.2.

This point of view upon categories can be illustrated by several functors of composite graphs. Such illustrations will get a precise meaning in section 4.

For instance, the property "each consecutive pair of arrows has a composite" can be illustrated by the inclusion functor:

\[
\begin{array}{ccc}
A_1 & \xrightarrow{a_1} & A_2 \\
\downarrow a_2 & & \downarrow a_3 \\
A_3 & \xrightarrow{a} & A_3
\end{array}
\]

The associativity property can be illustrated by the following functor, which maps both \((a_3 \circ a_2) \circ a_1\) and \(a_3 \circ (a_2 \circ a_1)\) to \(a_3 \circ a_2 \circ a_1\):

\[
\begin{array}{ccc}
A_1 & \xrightarrow{a_1} & A_2 \\
\downarrow a_2 & & \downarrow a_3 \\
A_3 & \xrightarrow{(a_3 \circ a_2) \circ a_1} & A_4 \\
\downarrow a_3 \circ a_2 \circ a_1 & & \downarrow a_3
\end{array}
\]

In these illustrations, the first composite graph represents the hypothesis \( H \) of the property, while the second composite graph represents its conclusion \( C \). The functor represents the deduction rule \( \frac{H}{C} \), i.e. "if \( H \) then \( C \)".

**Example 2.5.2**

We have just defined the category \( \text{Comp} \) of composite graphs and functors between composite graphs.

The functor \( U_{\mathcal{G}, \text{Comp}} = \text{Supp} : \text{Comp} \to \mathcal{G} \) maps each composite graph to its underlying graph and each functor to its underlying graph homomorphism. It has a left adjoint, which is the inclusion \( F_{\mathcal{G}, \text{Comp}} : \mathcal{G} \subseteq \text{Comp} \): it maps a graph \( \mathcal{G} \) to the composite graph with \( \mathcal{G} \) as its underlying graph and with no identity and no composite; this functor identifies \( \mathcal{G} \) with a full subcategory of \( \text{Comp} \). The functor \( U_{\mathcal{G}, \text{Comp}} \) is faithful, but it is not full; its left adjoint \( F_{\mathcal{G}, \text{Comp}} \) is full and faithful.

There is an inclusion functor \( U_{\text{Comp}, \mathcal{C}at} : \mathcal{C}at \subseteq \text{Comp} \), since the category \( \mathcal{C}at \) has just been identified to a full subcategory of \( \text{Comp} \). It has a left adjoint \( F_{\text{Comp}, \mathcal{C}at} : \text{Comp} \to \mathcal{C}at \): for all composite graph \( \mathcal{G} \), in order to get the category \( F_{\text{Comp}, \mathcal{C}at}(\mathcal{G}) \), we have to add the missing identities and composites, and to perform identifications, so that unitarity and associativity are satisfied. The functor \( U_{\text{Comp}, \mathcal{C}at} \) is full and faithful; its left adjoint \( F_{\text{Comp}, \mathcal{C}at} \) is neither full nor faithful.
Obviously $U_{\mathcal{G}, \mathsf{cat}} = U_{\mathcal{G}, \mathsf{comp}} \circ U_{\mathsf{comp}, \mathsf{cat}}$ and $F_{\mathcal{G}, \mathsf{cat}} = F_{\mathsf{comp}, \mathsf{cat}} \circ F_{\mathcal{G}, \mathsf{comp}}$.

![Diagram](attachment:image.png)

### 2.6 Limits

Let $\mathcal{I}$ be a compositio graph. The *typical $\mathcal{I}$-projective cone* is the compositio graph $\mathcal{C}_{pr} (\mathcal{I})$ made of $\mathcal{I}$, a point $V$, an arrow $pr_I : V \to I$ for all point $I$ of $\mathcal{I}$, such that $i \circ pr_I = pr_I$, for all arrow $i : I \to I'$ of $\mathcal{I}$. The inclusion functor is denoted $B_{\mathcal{I}} : \mathcal{I} \subseteq \mathcal{C}_{pr} (\mathcal{I})$.

A *$\mathcal{I}$-projective cone* in a compositio graph $\mathcal{G}$ is a functor $C : \mathcal{C}_{pr} (\mathcal{I}) \to \mathcal{G}$. Then the functor $B = C \circ B_{\mathcal{I}} : \mathcal{I} \to \mathcal{G}$ is the base of the projective cone $C$, the point $C(V)$ is its vertex, and the arrows $C(pr_I)$ are its projections.

**Definition 2.6.1**

A $\mathcal{I}$-projective cone $L$ in a category $\mathcal{A}$ is a *limit projective cone* if for all $C : \mathcal{C}_{pr} (\mathcal{I}) \to \mathcal{A}$ there is a unique projective factorisation arrow $\text{projact}_{C,L} : C(V) \to L(V)$ in $\mathcal{A}$ such that $L(pr_I) \circ \text{projact}_{C,L} = C(pr_I)$ for all point $I$ of $\mathcal{I}$.

![Diagram](attachment:image.png)

All the limit projective cones in $\mathcal{A}$ with the same base $B$ are isomorphic. When one of them is chosen, it is denoted $\text{projlim}(B)$ or $\text{projlim}_{I \in \mathcal{I}}(A_I)$ where $A_I = B(I)$.

A category $\mathcal{A}$ is *$\mathcal{I}$-complete* if each base $B : \mathcal{I} \to \mathcal{A}$ has a projective limit in $\mathcal{A}$. A category $\mathcal{A}$ is with chosen $\mathcal{I}$-projective limits if each base $B : \mathcal{I} \to \mathcal{A}$ has a chosen projective limit in $\mathcal{A}$.

When $\mathcal{I}$ is empty, then the vertex $L(V) = \mathbb{I}$ of $L$ is a *terminal point* of $\mathcal{A}$.

When $\mathcal{I}$ is discrete (i.e. without any arrow), then $L$ is the *product* of $B$ in $\mathcal{A}$, with vertex $L(V) = \prod_{I \in \mathcal{I}} B(I)$, or $L(V) = B(I_1) \times \cdots \times B(I_n)$ when $\mathcal{I} = \{I_1, \ldots, I_n\}$.

When $\mathcal{I}$ is $I_1 \xrightarrow{i_1} I, I_2 \xrightarrow{i_2} I_2$, then $L$ is the *pullback* of $B$ in $\mathcal{A}$, with vertex $L(V)$, sometimes written $L(V) = B(i_1) \times_{B(I_1)} B(I_2)$.

Among pullbacks, let $a : A_1 \to A$ be an arrow in $\mathcal{A}$, and let $B(I_1) = B(I_2) = A_1, B(I) = A$ and $B(i_1) = B(i_2) = a$. Then, $a$ is a monomorphism if and only if one of the projections $L(V) \to A_1$ is an isomorphism.

When $\mathcal{I}$ is $I_1 \xrightarrow{i_1} I, I_1 \xrightarrow{i_2} I$, then $L$ is the *equalizer* of $i_1$ and $i_2$ in $\mathcal{A}$.

By reversing the direction of the arrows in the cones, we get the *dual* notions.

Let $\mathcal{I}$ be a compositio graph. The *typical $\mathcal{I}$-inductive cone* is the compositio graph $\mathcal{C}_{in} (\mathcal{I})$ made of $\mathcal{I}$, a point $V$, an arrow $in_I : I \to V$ for all point $I$ of $\mathcal{I}$, such that $i \circ in_I = in_I$, for all arrow $i : I \to I'$ of $\mathcal{I}$. The inclusion functor is denoted $B_{\mathcal{I}} : \mathcal{I} \subseteq \mathcal{C}_{in} (\mathcal{I})$.

A *$\mathcal{I}$-inductive cone* in $\mathcal{G}$ is a functor $C : \mathcal{C}_{in} (\mathcal{I}) \to \mathcal{G}$, with its base $B = C \circ B_{\mathcal{I}} : \mathcal{I} \to \mathcal{G}$, its vertex $C(V)$ and its inductions $C(in_I)$.

**Definition 2.6.2**

A $\mathcal{I}$-inductive cone $L$ in $\mathcal{A}$ is a *limit inductive cone* if for all $C : \mathcal{C}_{in} (\mathcal{I}) \to \mathcal{A}$ there is a unique inductive
factorisation arrow \( \text{indfact}_{L,C} : L(V) \to C(V) \) such that \( \text{indfact}_{L,C} \circ L(\text{in}_I) = C(\text{in}_I) \) for all point \( I \) of \( \mathcal{I} \).

![Diagram](image)

All the limit inductive cones in \( \mathcal{A} \) with the same base \( B \) are isomorphic. When one of them is chosen, it is denoted \( \text{indlim}(B) \) or \( \text{indlim}_{\mathcal{I} \in \mathcal{A}}(A_I) \) where \( A_I = B(I) \).

A category \( \mathcal{A} \) is \( \mathcal{I} \)-cocomplete if each base \( B : \mathcal{I} \to \mathcal{A} \) has an inductive limit in \( \mathcal{A} \). A category \( \mathcal{A} \) is with chosen \( \mathcal{I} \)-inductive limits if each base \( B : \mathcal{I} \to \mathcal{A} \) has a chosen inductive limit in \( \mathcal{A} \).

The dual of a terminal point is an initial point, the dual of a product is a sum, the dual of a pullback is pushout, and the dual of an equalizer is a coequalizer.

The first following result is obvious, the second one is proven in [Mac Lane, 1971, p. 114].

**Proposition 2.6.3** (arrows on limits)

Let \( \mathcal{A} \) be a category and \( A \) a point of \( \mathcal{A} \).
- The functor \( \text{Hom}_\mathcal{A}(A, -) : \mathcal{A} \to \text{Set} \) maps projective limits to projective limits.
- The contravariant functor \( \text{Hom}_\mathcal{A}(-, A) : \mathcal{A} \to \text{Set} \) maps inductive limits to projective limits.

**Proposition 2.6.4** (adjoints on limits)

Let \( (F, U) \) be an adjunction. Then the functor \( U \) preserves the projective limits, and the functor \( F \) preserves the inductive limits.

**Example 2.6.5**

The category \( \text{Set} \) is \( \mathcal{I} \)-complete and \( \mathcal{I} \)-cocomplete for all (sufficiently small) \( \mathcal{I} \). The usual way to build projective and inductive limits via cartesian products, disjoint unions and quotients, yields a choice of limits and colimits.

In the category \( \text{Set} \), a terminal point is a one-element set, a product is a cartesian product, and a monomorphism is an injection. The initial point is the empty set, a sum is a disjoint union, and an epimorphism is a surjection.

# 3 Projective sketches, propagators, realizations

Basic notions about projective sketches are presented here. We define projective sketches and their homomorphisms, which we call propagators, as well as the category of realizations of a projective sketch. We state the fundamental theorem about the freely generated realization, which associates an adjunction \( (F_P, U_P) \) to each propagator \( P \). These notions are rather well known, from Ehresmann’s pioneering work in the 1960’s [Ehresmann, 1966]. Some of these notions can be found in [Coppey and Lair, 1984] and [Coppey and Lair, 1988], others in [Duval and Lair, 2001]. The fundamental theorem 3.4.1 is known as the associated sheaf theorem.

## 3.1 Projective sketches

**Definition 3.1.1**

A projective sketch \( \mathcal{E} \) is made of a composite graph \( \text{Supp}(\mathcal{E}) \), called the support of \( \mathcal{E} \), where some projective cones are called distinguished projective cones (or dpcs).

Let \( \mathcal{E} \) be a projective sketch.

A (potential) isomorphism is an arrow \( e_1 : E_1 \to E_2 \) with a (potential) inverse, i.e. such that there are
an arrow \( e_2 : E_2 \rightarrow E_1 \), two identities \( id_{E_1} \) and \( id_{E_2} \), and two composites \( e_1 \circ e_2 = id_{E_2} \) and \( e_2 \circ e_1 = id_{E_1} \). A (potential) monomorphism is an arrow \( e_1 : E_1 \rightarrow E_2 \) such that there is a distinguished projective cone with base \( E_1 \xrightarrow{e_1} E_2 \xleftarrow{\ell} E_1 \), and one of the projections from the vertex of this cone to \( E_1 \) is a potential isomorphic arrow.

A (potential) split monomorphism is an arrow \( e_1 : E_1 \rightarrow E_2 \) with a (potential) left inverse, i.e. such that there are an arrow \( e_2 : E_2 \rightarrow E_1 \), an identity \( id_{E_1} \), and a composite \( e_2 \circ e_1 = id_{E_1} \).

A (potential) split epimorphism is an arrow \( e_1 : E_1 \rightarrow E_2 \) with a (potential) right inverse, i.e. such that there are an arrow \( e_2 : E_2 \rightarrow E_1 \), an identity \( id_{E_1} \), and a composite \( e_1 \circ e_2 = id_{E_2} \).

A (potential) factorization arrow is an arrow \( \text{fact}_{C,L} : C \rightarrow L \) where \( C \) and \( L \) are projective cones with the same base \( B : I \rightarrow \text{Supp}(\mathcal{E}) \), with \( L \) distinguished, together with the composites \( L(pr_I) \circ \text{fact}_{C,L} = C(pr_I) \) for all point \( I \) of \( I \).

A (potential) terminal point is a point \( U \) together with a distinguished projective cone with empty base and vertex \( U \); this is denoted \( U = \emptyset \). Then, for each point \( E \) of \( \mathcal{E} \), there may be a potential factorization arrow \( \text{fact}_{E,U} : E \rightarrow U \).

By adding distinguished inductive cones, in a dual way, we get the (mixed) sketches, which will not play any important role in this paper. In mixed sketches, we could define a (potential) epimorphism and a (potential) initial point.

The generalization of this paper to mixed sketches would be far from trivial. It should use results from [Guitart and Lair, 1980] in order to generalize the freely generated realization theorem 3.4.1.

In this paper, we illustrate a projective sketch as its underlying compositive graph, together with the symbols \( \overline{\longrightarrow} \) for the projection arrows and \( \overline{\longleftarrow} \) for potential monomorphisms. There is a lot of ambiguity in such an illustration, which has to come with some additional information about the distinguished projective cones. The representation of composite projections may be omitted.

**Example 3.1.2**

Here are two projective sketches \( \mathcal{E}_{Ps} \) and \( \mathcal{E}_{Ar} \) without any distinguished projective cone. As will be seen in section 3.3, the names \( Pt \), \( Ar \), \( sce \) and \( tgt \) stand respectively for points, arrows, source and target.

\[
\mathcal{E}_{Ps} : \quad \begin{array}{c}
\text{Pt}
\end{array}
\]

\[
\mathcal{E}_{Ar} : \quad \begin{array}{c}
\text{Pt} \overset{sce}{\longrightarrow} \text{Ar} \overset{tgt}{\longleftarrow}
\end{array}
\]

Here is a projective sketch \( \mathcal{E}_{Gr}' \), with three distinguished projective cones. As will be seen in section 3.3, the names \( Lo \), \( Co \) and \( Tr \) stand respectively for loops, consecutive arrows and triangles.

\[
\mathcal{E}_{Gr}' : \quad \begin{array}{c}
\text{Pt} \overset{sce}{\longrightarrow} \text{Ar} \overset{tgt}{\longleftarrow}
\end{array}
\]

3.2 Propagators

**Definition 3.2.1**

Let \( \mathcal{E} \) and \( \mathcal{E}' \) be two projective sketches. A propagator \( P : \mathcal{E} \rightarrow \mathcal{E}' \) is a functor \( \text{Supp}(P) : \text{Supp}(\mathcal{E}) \rightarrow \text{Supp}(\mathcal{E}') \) which preserves the distinguished projective cones.

Obviously, up to size issues, the projective sketches and their propagators form a category Sketch.

An inclusion of projective sketches \( \mathcal{E} \subseteq \mathcal{E}' \) is a propagator \( P : \mathcal{E} \rightarrow \mathcal{E}' \) such that \( \text{Supp}(P) \) is an inclusion of compositive graphs.

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Example 3.2.2
\( \mathcal{E}_{\mathcal{S}t} \subseteq \mathcal{E}_{\mathcal{R}} \subseteq \mathcal{E}'_{\mathcal{R}} \).

3.3 Realizations

Definition 3.3.1
Let \( \mathcal{E} \) be a projective sketch and \( \mathcal{A} \) a category. A realization \( S : \mathcal{E} \rightarrow \mathcal{A} \) of \( \mathcal{E} \) with values in \( \mathcal{A} \) is a functor \( \text{Supp}(S) : \text{Supp}(\mathcal{E}) \rightarrow \mathcal{A} \) which maps each distinguished projective cone in \( \mathcal{E} \) to a limit projective cone in \( \mathcal{A} \).

So, a realization of \( \mathcal{E} \) maps a potential isomorphism (resp. monomorphism, split monomorphism, split epimorphism) of \( \mathcal{E} \) to a (real) isomorphism (resp. monomorphism, split monomorphism, split epimorphism) of \( \mathcal{A} \).

The category \( \mathcal{A} \) can be considered as a projective sketch: its support is the underlying compositive graph, and its dpcs are all its projective limit cones (with some care about the size of the indexations of the cones). Then, a realization of \( \mathcal{E} \) with values in \( \mathcal{A} \) is a propagator from \( \mathcal{E} \) to the projective sketch \( \mathcal{A} \).

Definition 3.3.2
Let \( S_1 \) and \( S_2 \) be two realizations of \( \mathcal{E} \) with values in \( \mathcal{A} \). A morphism \( \sigma : S_1 \rightarrow S_2 \) is a natural transformation between the underlying functors.

Obviously, the realizations of \( \mathcal{E} \) with values in \( \mathcal{A} \) and their morphisms form a category \( \text{Real}(\mathcal{E}, \mathcal{A}) \). In addition, for each point \( E \) of \( \mathcal{E} \), there is a functor \( \text{ev}_E : \text{Real}(\mathcal{E}, \mathcal{A}) \rightarrow \mathcal{A} \), called the evaluation at \( E \), such that \( \text{ev}_E(S) = S(E) \) for all realization and \( \text{ev}_E(\sigma) = \sigma(E) \) for all morphism of realizations.

In addition, for all propagator \( P : \mathcal{E} \rightarrow \mathcal{E}' \) there is a functor \( \text{Real}(P, \mathcal{A}) : \text{Real}(\mathcal{E}', \mathcal{A}) \rightarrow \text{Real}(\mathcal{E}, \mathcal{A}) \), which maps all realization \( S' \) of \( \mathcal{E}' \) to the realization \( S' \circ P \) of \( \mathcal{E} \), and all morphism of realizations \( \sigma' : S'_1 \rightarrow S'_2 \) of \( \mathcal{E}' \) to the morphism of realizations \( \sigma' \circ P : S'_1 \circ P \rightarrow S'_2 \circ P \) of \( \mathcal{E} \). Altogether, we get a contravariant functor:

\[ \text{Real}(-, \mathcal{A}) : Sketch \rightarrow \text{Cat}. \]

Proposition 3.3.3
The functor \( \text{Real}(-, \mathcal{A}) \) maps inductive limits to projective limits.

A contravariant realization \( Z : \mathcal{E} \rightarrow \mathcal{A} \) of \( \mathcal{E} \) with values in a category \( \mathcal{A} \) is a contravariant functor \( \text{Supp}(Z) : \text{Supp}(\mathcal{E}) \rightarrow \mathcal{A} \) which maps each distinguished projective cone in \( \mathcal{E} \) to a limit inductive cone in \( \mathcal{A} \).

Example 3.3.4
A realization \( S \) of \( \mathcal{E}_{\mathcal{S}t} \) is a set \( S(\text{Pt}) \), and a morphism \( \sigma : S_1 \rightarrow S_2 \) is a map \( \sigma(\text{Pt}) : S_1(\text{Pt}) \rightarrow S_2(\text{Pt}) \).

So, there is an isomorphism \( \text{Real}(\mathcal{E}_{\mathcal{S}t}) \cong \mathcal{S}et \).

A realization \( S \) of \( \mathcal{E}_{\mathcal{R}} \) is made of two sets \( S(\text{Pt}) \) and \( S(\text{Ar}) \), and two maps \( S(\text{src}), S(\text{tgt}) : S(\text{Ar}) \rightarrow S(\text{Pt}) \): it is a directed graph. And indeed, there is an isomorphism \( \text{Real}(\mathcal{E}_{\mathcal{R}}) \cong \mathcal{G}r \).

There is an equivalence \( \text{Real}(\mathcal{E}'_{\mathcal{R}}) \cong \mathcal{G}r \). Indeed, a realization \( S \) of \( \mathcal{E}'_{\mathcal{R}} \) is a directed graph, together with sets \( S(\text{Lo}), S(\text{Co}) \) and \( S(\text{Tr}) \) which are, because of the distinguished projective cones, isomorphic to, respectively, the set of loops, the set of consecutive arrows, and the set of triangles, of this directed graph.

3.4 Adjunction

The category of set-valued realizations of \( \mathcal{E} \) is \( \text{Real}(\mathcal{E}) = \mathcal{R}eal(\mathcal{E}, \mathcal{S}et) \). Up to some care about size issues, the category \( \text{Real}(\mathcal{E}) \) is both complete and cocomplete.
To each propagator $P : \mathcal{E} \to \mathcal{E}'$ is associated the underlying functor:

$$U_P = \mathsf{Real}(P) : \mathsf{Real}(\mathcal{E}') \to \mathsf{Real}(\mathcal{E}).$$

The following result is a fundamental one, it is known as the associated sheaf theorem. A proof can be found in [Duval and Lair, 2001]. The generalization of this result to mixed sketches, which is far from trivial, is done in [Guitart and Lair, 1980].

**Theorem 3.4.1 (freely generated realization)**

Let $P : \mathcal{E} \to \mathcal{E}'$ be a propagator. The functor $U_P : \mathsf{Real}(\mathcal{E}') \to \mathsf{Real}(\mathcal{E})$ has a left adjoint:

$$F_P : \mathsf{Real}(\mathcal{E}) \to \mathsf{Real}(\mathcal{E}').$$

The functor $F_P$ is the freely generating functor associated to $P$.

From the definition of an adjunction, it follows that, for all realizations $S$ of $\mathcal{E}$ and $S'$ of $\mathcal{E}'$, there is a bijection, which is natural in $S$ and in $S'$:

$$\text{Hom}_{\mathsf{Real}(\mathcal{E})}(S, U_P(S')) \cong \text{Hom}_{\mathsf{Real}(\mathcal{E}')} (F_P(S), S').$$

The corresponding monad and comonad are respectively denoted (the subscript $P$ may be omitted): $(M_P : \mathsf{Real}(\mathcal{E}) \to \mathsf{Real}(\mathcal{E})$, $\eta_P : \mathsf{id}_{\mathsf{Real}(\mathcal{E})} \Rightarrow M_P$, $\mu_P : M_P^2 \Rightarrow M_P)$ and $\varepsilon_P : F_P \circ U_P \Rightarrow \mathsf{id}_{\mathsf{Real}(\mathcal{E})}$.

**Proposition 3.4.2**

Let $P_1 : \mathcal{E}_1 \to \mathcal{E}_1'$, $P_2 : \mathcal{E}_2 \to \mathcal{E}_2'$, $T_y : \mathcal{E}_1 \to \mathcal{E}_2$ and $T'_y : \mathcal{E}_1' \to \mathcal{E}_2'$ be a commutative square in the category of projective sketches. Then, there is a natural transformation:

$$(F_{P_2} \circ \varepsilon_{T'_y})_* : F_{P_1} \circ U_{T_y} \Rightarrow U_{T'_y} \circ F_{P_2} : \mathsf{Real}(\mathcal{E}_2) \to \mathsf{Real}(\mathcal{E}_2').$$

This natural transformation is not, in general, a natural isomorphism.

**Proof.** From the counit $\varepsilon_L : F_L \circ U_L \Rightarrow \mathsf{id}_{\mathsf{Real}(\mathcal{E}_2)}$, we get the natural transformation $F_{P_2} \circ \varepsilon_L : F_{P_2} \circ F_L \circ U_L \Rightarrow F_{P_2} : \mathsf{Real}(\mathcal{E}_2) \to \mathsf{Real}(\mathcal{E}_2')$. Since $P_2 \circ L = L' \circ P_1$, the previous result can be written as $F_{P_2} \circ \varepsilon_L : F_L \circ \varepsilon_{P_2} \circ U_L \Rightarrow F_{P_2}$. So, by adjunction, we get the proposition. $\square$

It follows that, for all realization $S_2$ of $\mathcal{E}_2$, there is a morphism $F_{P_1}(U_L(S_2)) \to U_{L'}(F_{P_2}(S_2))$ in $\mathsf{Real}(\mathcal{E}_2')$.

**Example 3.4.3**

Let $P$ denote the inclusion $P : \mathcal{E}_{\text{Set}} \subseteq \mathcal{G}_{\text{Gr}}$. The underlying functor $U_P : \mathsf{Real}(\mathcal{G}_{\text{Gr}}) \to \mathsf{Real}(\mathcal{E}_{\text{Set}})$ is the functor $Pt : \mathcal{G}_{\text{Gr}} \to \mathsf{Set}$, from section 2.2, which forgets the arrows. The freely generating functor $F_P : \mathsf{Real}(\mathcal{E}_{\text{Set}}) \to \mathsf{Real}(\mathcal{G}_{\text{Gr}})$ is the inclusion functor $\mathsf{Set} \subseteq \mathcal{G}_{\text{Gr}}$ from section 2.2.

### 3.5 Equivalence of sketches.

The following definition of conservative propagators is semantic: it is relative to the set-valued realizations of the sketches involved.

**Definition 3.5.1**

A propagator $Q : \mathcal{E} \to \mathcal{E}'$ is conservative if both functors $F_Q$ and $U_Q$ are full and faithful.
From theorem 2.3.4, \(Q\) is conservative if and only if the unit \(\eta_Q\) and the counit \(\varepsilon_Q\) are natural isomorphisms.

**Definition 3.5.2**
The equivalence of projective sketches is the equivalence relation generated by:
- \(\mathcal{E} \equiv \mathcal{E}'\) as soon as there is a conservative propagator from \(\mathcal{E}\) to \(\mathcal{E}'\).

A zig-zag of propagators \((P_1, \ldots, P_n)\) from \(\mathcal{E}\) to \(\mathcal{E}'\) is made of projective sketches \(\mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_n\) such that \(\mathcal{E}_0 = \mathcal{E}\) and \(\mathcal{E}_n = \mathcal{E}'\), and of propagators \(P_1, \ldots, P_n\) with, for each \(k\) from 1 to \(n\), either \(P_k : \mathcal{E}_{k-1} \to \mathcal{E}_k\) or \(P_k : \mathcal{E}_k \to \mathcal{E}_{k-1}\). Then, clearly, two projective sketches \(\mathcal{E}\) and \(\mathcal{E}'\) are equivalent if there is a zig-zag of conservative propagators from \(\mathcal{E}\) to \(\mathcal{E}'\).

From theorem 2.3.4, if two projective sketches \(\mathcal{E}\) and \(\mathcal{E}'\) are equivalent then the categories \(\text{Real}(\mathcal{E})\) and \(\text{Real}(\mathcal{E}')\) are equivalent: if \(\mathcal{E} \equiv \mathcal{E}'\) then \(\text{Real}(\mathcal{E}) \simeq \text{Real}(\mathcal{E}')\).

In the following result are listed some families of conservative propagators, which can be composed or used in zig-zag in order to get equivalences of projective sketches. There are many other ways to get conservative propagators and equivalences of projective sketches.

**Proposition 3.5.3 (construction of conservative propagators)**
Let \(Q : \mathcal{E} \to \mathcal{E}'\) be a propagator such that, either:
- \(Q\) adds an identity loop at a point of \(\mathcal{E}\),
- \(Q\) adds a composite for a pair of consecutive arrows of \(\mathcal{E}\),
- \(Q\) adds a distinguished projective cone for a base in \(\mathcal{E}\),
- \(Q\) adds a potential factorization arrow, or identifies two potential factorisation arrows, between a projective cone and a distinguished projective cone with the same base, both in \(\mathcal{E}\),
- \(Q\) states that an invertible arrow or an identity arrow is a monomorphic arrow.
- \(Q\) adds a new point \(E'\), the identities \(id_E\) (if it is not yet in \(\mathcal{E}\) and \(id_{E'}\), two arrows \(e_1 : E \to E'\) and \(e_2 : E' \to E\) with the composites \(e_2 \circ e_1 = id_E\) and \(e_1 \circ e_2 = id_{E'}\).
- \(Q\) maps an invertible arrow \(e : E_1 \to E_2\), with \(E_1 \neq E_2\), towards an identity arrow.

Then \(Q\) is a conservative propagator.

**Proof.** This result is easily derived from the properties of the complete category \(\text{Set}\); for instance the image of a point of \(\mathcal{E}\) is a point in \(\text{Set}\), so that it has one identity arrow, and so on. \(\Box\)

On the contrary, a propagator which maps an invertible arrow \(e : E \to E\) towards an identity arrow is not conservative, in general. Indeed, let \(\mathcal{E}\) be made of one point \(E\), the identity \(id_E\), and two arrows \(e_1, e_2 : E \to E\) with the composites \(e_2 \circ e_1 = id_E\) and \(e_1 \circ e_2 = id_E\). Let \(\mathcal{E}'\) be made of one point \(E'\) and the identity \(id_{E'}\), and let \(P : \mathcal{E} \to \mathcal{E}'\) be the unique propagator from \(\mathcal{E}\) to \(\mathcal{E}'\). Now, let \(S\) be a realization of \(\mathcal{E}\) such that \(S(E)\) has two elements \(x\) and \(y\), and \(S(e_1) = S(e_2)\) permutes \(x\) and \(y\). Then \(F_{P}(S)\) identifies \(x\) and \(y\), so that \(M_{P}(S)(E)\) is made of only one element, and \(\eta_{P\cdot S}\) cannot be an isomorphism.

**Definition 3.5.4**
The equivalence of propagators is the equivalence relation \(P \equiv P'\) (where \(P : \mathcal{E}_1 \to \mathcal{E}_2\) and \(P' : \mathcal{E}'_1 \to \mathcal{E}'_2\)) generated by:
- \(P \equiv P'\) as soon as \(\mathcal{E}_2 = \mathcal{E}'_2\) and there is a conservative propagator \(Q_1 : \mathcal{E}_1 \to \mathcal{E}'_1\) such that \(P' \circ Q_1 = P\),
- \(P \equiv P'\) as soon as \(\mathcal{E}_1 = \mathcal{E}'_1\) and there is a conservative propagator \(Q_2 : \mathcal{E}_2 \to \mathcal{E}'_2\) such that \(Q_2 \circ P = P'\).

If \(P \equiv P'\), then clearly \(\mathcal{E}_1 \equiv \mathcal{E}'_1\) and \(\mathcal{E}_2 \equiv \mathcal{E}'_2\).

**Example 3.5.5**
The inclusion of \(\mathcal{E}_{Gr}\) in \(\mathcal{E}'_{Gr}\) (from section 3.1) is a conservative propagator; indeed, it may easily be decomposed in several steps, which are either the addition of a distinguished projective cone for a given base, or the addition of a composite for a pair of consecutive arrows. In this way, from the isomorphism \(\text{Real}(\mathcal{E}_{Gr}) \cong Gr\), we get another proof of the equivalence \(\text{Real}(\mathcal{E}'_{Gr}) \simeq Gr\).
3.6 Prototypes and types

Definition 3.6.1
A \textit{projective prototype} is a projective sketch such that its support is a category and its distinguished projective cones are limit cones.

It can be proven that each projective sketch $\mathcal{E}$ freely generates a projective prototype $Pr(\mathcal{E})$. The unit propagator $\mathcal{E} \rightarrow Pr(\mathcal{E})$ maps each distinguished projective cone of $\mathcal{E}$ to a distinguished limit projective cone of $Pr(\mathcal{E})$. It follows that $Real(Pr(\mathcal{E})) \cong Real(\mathcal{E})$.

Definition 3.6.2
With respect to some family of compositive graphs for indexations, a \textit{projective type} is a category with with chosen projective limit cones (as defined in section 2.6).

A projective type can be considered as a projective prototype, by distinguishing all its chosen projective cones.

It can be proven that each projective sketch $\mathcal{E}$ freely generates a projective type $Ty(\mathcal{E})$. The unit propagator $\mathcal{E} \rightarrow Ty(\mathcal{E})$ maps each distinguished projective cone of $\mathcal{E}$ to a chosen (hence distinguished) limit projective cone of $Ty(\mathcal{E})$. It follows that $Real(Ty(\mathcal{E})) \cong Real(\mathcal{E})$.

Usually, the same notation is used for the points and arrows of $\mathcal{E}$ and their images in $Pr(\mathcal{E})$ and in $Ty(\mathcal{E})$, although the unit propagators $\mathcal{E} \rightarrow Pr(\mathcal{E})$ and $\mathcal{E} \rightarrow Ty(\mathcal{E})$ need not be inclusions.

3.7 Yoneda lemma for projective sketches

Let $\mathcal{E}$ be a projective sketch, then from section 2.4, there is a Yoneda contravariant functor:

$$Y_{Pr(\mathcal{E})} : Pr(\mathcal{E}) \leftrightarrow Func(Pr(\mathcal{E}), \mathcal{S}).$$

From proposition 2.6.3, the functor $Hom_{Pr(\mathcal{E})}(E, -) : Pr(\mathcal{E}) \rightarrow \mathcal{S}$ maps projective limits to projective limits. So, the functor $Y_{Pr(\mathcal{E})}(E) : Pr(\mathcal{E}) \rightarrow \mathcal{S}$ preserves the projective limit cones, which means that the image of $Y_{Pr(\mathcal{E})}$ is contained in $Real(Pr(\mathcal{E}))$:

$$Y_{Pr(\mathcal{E})} : Pr(\mathcal{E}) \leftrightarrow Real(Pr(\mathcal{E})).$$

In addition, since $Real(Pr(\mathcal{E}))$ is isomorphic to $Real(\mathcal{E})$, by composition of $Y_{Pr(\mathcal{E})}$ with the unit propagator $\mathcal{E} \rightarrow Pr(\mathcal{E})$, we get a contravariant functor:

$$Y_{\mathcal{E}} : \mathcal{E} \leftrightarrow Real(\mathcal{E}).$$

From proposition 2.6.3, the contravariant functor $Hom_{Real(\mathcal{E})}(-, S) : Real(\mathcal{E}) \leftrightarrow \mathcal{S}$ maps inductive limits to projective limits. So, the functor $Y_{\mathcal{E}}$ maps distinguished projective cones to limit inductive cones, which means that it is a contravariant realization of $\mathcal{E}$.

Theorem 3.7.1 (Yoneda lemma for projective sketches)

The Yoneda contravariant realization $Y_{\mathcal{E}} : \mathcal{E} \leftrightarrow Real(\mathcal{E})$ is such that, for each point $E$ of $\mathcal{E}$ and each realization $S$ of $\mathcal{E}$, naturally in $E$ and in $S$, the map $\sigma \mapsto \sigma_E(id_E)$ is a bijection:

$$Hom_{Real(\mathcal{E})}(Y_{\mathcal{E}}(E), S) \xrightarrow{\cong} S(E).$$

A consequence of theorem 3.7.1 is the density result of corollary 3.7.3 below: any set-valued realization of $\mathcal{E}$ is the vertex of an inductive limit cone which has its base in $Y_{\mathcal{E}}(\mathcal{E})$. The description of this cone makes use of a blow-up of $Supp(\mathcal{E})$. 


Definition 3.7.2
Let $\mathcal{G}$ be a directed graph and $H : \mathcal{G} \to \mathbf{Set}$ a functor. The blow-up $\mathcal{G}\setminus H$ of $\mathcal{G}$ by $H$ is the directed graph with:
- a point $[G, z]$ for all point $G$ of $\mathcal{G}$ and all $x \in H(G)$,
- an arrow $[g, z] : [G, z] \to [G', z']$ for all arrow $g : G \to G'$ of $\mathcal{G}$ and all $x \in H(G)$, where $x' = H(g)(x)$,
- an identity $id_{[G, z]} = [id_G, z]$ for all identity $id_G$ of $\mathcal{G}$ and all $x \in H(G)$,
- a composite $[g_2, g_1, x_1] = [g_2, x_2] \circ [g_1, x_1]$ for all composite $g_2 \circ g_1$ of $\mathcal{G}$ and all $x_1 \in sce(g_1)$, where $x_2 = H(g_1)(x_1)$.

Let us write $Y$ for $Y_{\mathcal{E}}$. Let $S$ be a set-valued realization of $\mathcal{E}$, and $I = (\text{Supp}(\mathcal{E}) \setminus \text{Supp}(S))^{op}$. Let $C_S$ denote the $I$-inductive cone in $\mathbf{Real}(\mathcal{E})$ with:
- vertex $S$,
- base $B : I \to \mathbf{Real}(\mathcal{E})$ such that $B([E, z]) = Y(E)$ for all point $[E, z]$ of $I$ and $B([e, z]) = Y(e)$ for all arrow $[e, z]$ of $I$,
- inductions $in_{[E, z]} : Y(E) \to S$ such that for each point $E'$ in $\mathcal{E}$ the map $in_{[E', z]}(E') : \text{Hom}_{\mathbf{P}, \mathcal{E}}(E, E') \to S(E')$ maps $e$ towards $S(e)(z)$.

It is easy to check that this is indeed an inductive cone. The density of Yoneda realization states that it is an inductive limit cone.

Corollary 3.7.3 (density of Yoneda realization)
Let $S$ be a realization of $\mathcal{E}$. Then the inductive cone $C_S$ in $\mathbf{Real}(\mathcal{E})$ is a limit cone, which is written: $S \cong \text{indlim}_{\mathcal{E}}(Y_{\mathcal{E}}(E))$.

Proposition 3.7.4
Let $P : \mathcal{E} \to \mathcal{E}'$ be a propagator. Then there is an isomorphism of contravariant models of $\mathcal{E}$ with values in $\mathbf{Real}(\mathcal{E}')$: $F_P \circ Y_{\mathcal{E}} \cong Y_{\mathcal{E}'} \circ P$.

Proof. Let $E$ be a point of $\mathcal{E}$, and $S'$ a realization of $\mathcal{E}'$. Then, from Yoneda lemma applied to $\mathcal{E}$, $\text{Hom}_{\mathbf{Real}(\mathcal{E})}(Y_{\mathcal{E}}(E), U_P(S')) \cong U_P(S')(E) = S'(P(E))$. On the other hand, from Yoneda lemma applied to $\mathcal{E}'$, $\text{Hom}_{\mathbf{Real}(\mathcal{E}')} (Y_{\mathcal{E}'}(P(E)), S') \cong S'(P(E))$. So that:

$\text{Hom}_{\mathbf{Real}(\mathcal{E})}(Y_{\mathcal{E}}(E), U_P(S')) \cong \text{Hom}_{\mathbf{Real}(\mathcal{E}')} (Y_{\mathcal{E}'}(P(E)), S')$, naturally in $E$ and $S'$,

which means that $Y_{\mathcal{E}'} \circ P$ is isomorphic to $F_P \circ Y_{\mathcal{E}}$. \(\square\)

4 Fractioning and filling propagators

In this section, we focus on two families of propagators. A fractioning propagator $K$ is such that $U_K$ is full and faithful, while a filling propagator $J$ is such that $F_J$ is full and faithful. We prove that any propagator $P$ can be decomposed as $P = K \circ J$ with $K$ fractioning and $J$ filling. The words “fractioning” and “filling” stem from theorems 4.2.2 and 4.3.2, respectively.

4.1 A basic example

Directed graphs.

As in section 3.1, let $\mathcal{E}_{\mathcal{G}}$ be the following projective sketch (without any distinguished projective cone), such that $\mathbf{Real}(\mathcal{E}_{\mathcal{G}}) \cong \mathcal{G}$:

\[ \mathcal{E}_{\mathcal{G}} : \begin{array}{c}
\text{Pt} \\
\downarrow^{\text{tg}} \\
\text{Ar}
\end{array} \]

Let $\mathcal{E}'_{\mathcal{G}}$ denote the projective sketch described in section 3.1, together with the composite $p' = sce \circ p = tg \circ p : Lo \to Pt$. The precise description of its distinguished projective cones is given in section 3.1.
The inclusion of $\mathcal{E}_{gr}$ in $\mathcal{E}_{gr}'$ is conservative, so that $\mathcal{E}_{gr} \equiv \mathcal{E}_{gr}'$ and $\text{Real}(\mathcal{E}_{gr}') \simeq \mathcal{G}_r$.

Categories.
Let us add to $\mathcal{E}_{gr}'$:
- two identities $id_{C_0}$ and $id_{Pt}$ (not represented in the picture),
- two arrows $s : C_0 \to Tr$ and $s' : Pt \to Lo$ such that $r_1 \circ s = id_{C_0}$ and $p' \circ s' = id_{Pt}$,
- and whatever is needed to express the unitarity and associativity of categories.

It is easy to check that the resulting projective sketch $\mathcal{E}_{cat}$ is such that $\text{Real}(\mathcal{E}_{cat}) \simeq \text{Cat}$.

Let us add to $\mathcal{E}_{cat}$:
- two 2DCs such that the identity arrows $id_{C_0}$ and $id_{Pt}$ are potential monomorphisms,
- two composite arrows $\text{comp} = r_2 \circ s : C_0 \to Ar$, for the composition, and $\text{setid} = p \circ s' : Pt \to Ar$, for the selection of identities.

Then the inclusion of $\mathcal{E}_{cat}$ in $\mathcal{E}_{cat}'$ is conservative, so that $\mathcal{E}_{cat} \equiv \mathcal{E}_{cat}'$ and $\text{Real}(\mathcal{E}_{cat}') \simeq \text{Cat}$.

The following illustration does not represent the identities, nor the unitarity and associativity properties.

Let $P : \mathcal{E}_{gr} \to \mathcal{E}_{cat}$ be the inclusion. Let $\mathcal{G}$ be a graph, then the graph $UP(FP(\mathcal{G}))$ is not isomorphic to $\mathcal{G}$. Indeed, the freely generating functor $FP$ adds the required identities and composites, which are not removed by the underlying functor $UP$. So, the unit $\eta : \mathcal{G} \to UP(FP(\mathcal{G}))$ is far from an isomorphism. For instance:

Let $\mathcal{A}$ be a category, then the category $FP(UP(\mathcal{A}))$ is not isomorphic to $\mathcal{A}$. Indeed, the underlying functor $UP$ forgets that some arrows are identities or composites. Then, the freely generating functor $FP$ adds to the graph $UP(\mathcal{A})$ a new copy of these identities or composites. So, the counit $\varepsilon_\mathcal{A} : F(U(\mathcal{A})) \to \mathcal{A}$ is
far from an isomorphism. For instance:

\[
\begin{array}{c}
\begin{array}{c}
\drawDiagram{A_1}{A_2}{A_3}
\end{array}
\end{array}
\]

\textbf{Composite graphs.}

Let us add to \( \mathcal{E}_\mathcal{G} \):
- two points, \( \text{Comp} \) for the consecutive arrows with a composite, and \( \text{Ptid} \) for the points with an identity, together with arrows \( m' : \text{Ptid} \rightarrow \text{Pt} \) and \( m : \text{Comp} \rightarrow \text{Co} \) which are potential monomorphisms.
- two arrows \( s : \text{Comp} \rightarrow \text{Tr} \) and \( s' : \text{Ptid} \rightarrow \text{Lo} \) such that \( r_1 \circ s = m \) and \( p' \circ s' = m' \),
- two composite arrows \( \text{comp} = r_2 \circ s : \text{Comp} \rightarrow \text{Ar} \), for the composition, and \( \text{selid} = p \circ s' : \text{Ptid} \rightarrow \text{Ar} \), for the selection of identities.

It is easy to check that the resulting projective sketch \( \mathcal{E}_{\text{comp}} \) is such that \( \text{Real}(\mathcal{E}_{\text{comp}}) \simeq \text{Comp} \).

\[
\mathcal{E}_{\text{comp}}:
\]

\textbf{Decomposition of} \( P \).

The propagator \( P : \mathcal{E}_\mathcal{G} \rightarrow \mathcal{E}_{\text{Cat}} \) is equivalent to \( P' : \mathcal{E}_\mathcal{G} \rightarrow \mathcal{E}'_{\text{Cat}} \), which can be decomposed as \( P' = K' \circ J \)
where \( J : \mathcal{E}_\mathcal{G} \rightarrow \mathcal{E}_{\text{comp}} \) is the inclusion and \( K' : \mathcal{E}_{\text{comp}} \rightarrow \mathcal{E}'_{\text{Cat}} \) is such that \( m \) and \( m' \) are mapped to \( \text{id}_{\text{Co}} \) and \( \text{id}_{P} \), respectively.

Let \( \mathcal{G} \) be a graph, then the graph \( U_J(F_J(\mathcal{G})) \) is isomorphic to \( \mathcal{G} \), because the composite graph \( F_J(\mathcal{G}) \)
has neither identities nor composites.

Let \( \mathcal{A} \) be a category, then clearly the category \( F_{K'}(U_{K'}(\mathcal{A})) \) is isomorphic to \( \mathcal{A} \).

Now, let us come back to the propagator \( P : \mathcal{E}_\mathcal{G} \rightarrow \mathcal{E}_{\text{Cat}} \) and to the construction of the category \( F_P(\mathcal{G}) \)
which is freely generated by some given graph \( \mathcal{G} \). Up to equivalence, we can rather consider the propagator \( P' : \mathcal{E}_\mathcal{G} \rightarrow \mathcal{E}'_{\text{Cat}} \) and build the category \( F_{P'}(\mathcal{G}) \). The intermediate sketch \( \mathcal{E}_{\text{comp}} \) can be used in order to get a progressive construction of \( F_{P'}(\mathcal{G}) \). First, \( F_{P'}(\mathcal{G}) = F_{K'}(F_J(\mathcal{G})) \), where \( F_J(\mathcal{G}) \) is easily obtained: it is \( \mathcal{G} \) together with no identity and no composite. So, we can assume that \( \mathcal{G} \) is a compositive graph, and look for a progressive construction of \( F_{K'}(\mathcal{G}) \). If \( G \) is a point in \( \mathcal{G} \) without an identity, we can build a compositive graph by adding \( \text{id}_G : G \rightarrow G \). If \( g_1 : G_1 \rightarrow G_2 \) and \( g_2 : G_2 \rightarrow G_3 \) are successive arrows in \( \mathcal{G} \) without a composite, we can build a compositive graph by adding \( g_2 \circ g_1 : G_1 \rightarrow G_3 \). In both cases, the resulting compositive graph \( \mathcal{G}' \) is such that \( F_{K'}(\mathcal{G}) = F_{K'}(\mathcal{G}') \), so that the construction may start again.
from \( G' \).
So, the composites and identities can be built little by little, from a directed graph (where they are nowhere defined) to a category (where they are everywhere defined), thanks to intermediate compositive graphs (where they are partially defined).
In the following, we prove that this property of \( P : \mathcal{E}_{\Phi} \to \mathcal{E}_{\text{cat}} \) can be generalized to any propagator.

### 4.2 Fractioning propagators

**Definition 4.2.1**
A propagator \( K : \mathcal{E} \to \mathcal{F} \) is fractioning if the underlying functor \( U_K \) is full and faithful.

From theorem 2.3.4, \( K \) is fractioning if and only if the counit \( \varepsilon_K \) is a natural isomorphism:

\[
\varepsilon_K : F_K \circ U_K \cong id_{\text{Real}\{\mathcal{F}\}}.
\]

Then, the multiplication \( \mu_K \) is a natural isomorphism, i.e. the monad associated to \( K \) is idempotent:

\[
\mu_K : M_K \cong M_K.
\]

Obviously, a conservative propagator is fractioning, the composite of fractioning propagators is fractioning, and a propagator which is equivalent to a fractioning one is also fractioning.

On the other hand, we say that a propagator \( K : \mathcal{E} \to \mathcal{F} \) adds an inverse to an arrow \( e : E_1 \to E_2 \) of \( \mathcal{E} \) if it adds an arrow \( e^{-1} : E_2 \to E_1 \), two identities \( id_{E_1} \) and \( id_{E_2} \) if they are needed, and two composites \( e^{-1} \circ e = id_{E_1} \) and \( e \circ e^{-1} = id_{E_2} \).

**Theorem 4.2.2 (fractioning propagators)**
A propagator is fractioning if and only if, up to equivalence, it adds inverses to arrows.

**Proof (partial).** We only prove here the easy part of this result. Similar results can be found in [Gabriel and Zisman, 1967] and in [Hebert, Adamek and Rosicky, 2001].

Let us assume that \( K \) adds an inverse to an arrow \( e : E_1 \to E_2 \) of \( \mathcal{E} \). Let \( D \) be a realization of \( \mathcal{F} \), so that the map \( D(e^{-1}) \) is the inverse of \( D(e) \). In \( U(D) \), the map \( U(D)(e) \) is equal to \( D(e) \), so that it is invertible. Then, \( F(U(D)) \) only gives a name to the inverse of \( U(D)(e) \), so that \( \varepsilon(D) : F \circ U(D) \to D \) is an isomorphism. It follows that \( K \) is fractioning, so that any propagator which adds inverses to arrows is fractioning. \( \square \)

**Theorem 4.2.3**
A propagator is fractioning if and only if, up to equivalence, it consists in the distinction of projective cones.

**Proof.** By theorem 4.2.2, we have to prove that, up to equivalence, a propagator \( K \) adds inverses to arrows if and only if it distinguishes projective cones.

Let \( e : E_1 \to E_2 \) be an arrow in \( \mathcal{E} \), and let us distinguish the projective cone with vertex \( E_1 \), base \( E_2 \) and projection \( e \). Then, up to equivalence, we can add the factorization arrow \( f = \text{fact}(id_{E_2}, e) : E_2 \to E_1 \). The property of factorization arrows states that \( e \circ f = id_{E_2} \). It follows that \( e \circ (f \circ e) = (e \circ f) \circ e = e \), which means that \( f \circ e = \text{fact}(e, e) \), but clearly \( \text{fact}(e, e) = id_{E_1} \), so that the unicity of factorization arrows proves that \( f \circ e = id_{E_1} \). So, \( f \) is an inverse of \( e \).

Let \( C \) be a projective cone in \( \mathcal{E} \) with base \( B \) and vertex \( E_1 \). Then, up to equivalence, we can add a distinguished projective cone \( C' \) with the same base \( B \) and some vertex \( E_2 \), and the factorization arrow \( e = \text{fact}(C, C') : E_1 \to E_2 \). Let us add an inverse \( e^{-1} \) to \( e \). Then, up to equivalence, we can distinguish the cone \( C \). \( \square \)
Proposition 4.2.4
A propagator which maps an arrow to an identity is fractioning.

Proof. Let us assume that \( K : \mathcal{E} \rightarrow \mathcal{F} \) maps an arrow \( e : E_1 \rightarrow E_2 \) of \( \mathcal{E} \) to an identity \( id_{E'} : E' \rightarrow E' \) of \( \mathcal{F} \). Let \( D \) be a realization of \( \mathcal{F} \), so that the map \( D(K(e)) \) is the identity of \( D(E') \). In \( U(D) \), the sets \( U(D)(E_1) \) and \( U(D)(E_2) \) are both equal to \( D(E') \), and the map \( U(D)(e) \) is the identity. So, \( \varepsilon(D) : F \circ U(D) \rightarrow D \) is an isomorphism. It follows that \( K \) is fractioning. \( \square \)

Let \( e : E_1 \rightarrow E_2 \) be an arrow in a projective sketch \( \mathcal{E} \). A propagator \( P : \mathcal{E} \rightarrow \mathcal{E}' \) adds a restriction to \( e \) with respect to \( m_1 \) and \( m_2 \), where \( m_1 : E_1' \rightarrow E_1 \) and \( m_2 : E_2' \rightarrow E_2 \) are arrows of \( \mathcal{E} \) and \( m_2 \) is a potential monomorphism, if it adds an arrow \( e' : E_1' \rightarrow E_2' \) with a commutative square \( e \circ m_1 = m_2 \circ e' \).

Proposition 4.2.5
A propagator which adds a restriction to an arrow is fractioning.

Proof. Let us assume that \( K : \mathcal{E} \rightarrow \mathcal{F} \) adds a restriction \( e' : E_1' \rightarrow E_2' \) to an arrow \( e : E_1 \rightarrow E_2 \) with respect to \( m_1 \) and \( m_2 \). Let \( D \) be a realization of \( \mathcal{F} \), so that the map \( D(K(e')) : D(K(E_1')) \rightarrow D(K(E_2')) \) is the restriction of \( D(K(e)) \). In \( U(D) \), it remains true that \( U(D)(e) \circ U(D)(m_1) = U(D)(m_2) \circ f \) for some map \( f \). Since the map \( U(D)(m_2) \) is injective, the map \( f \) is characterized by this equality. So, \( F(U(D)) \) only gives the name \( F(U(D))(e') \) to the map \( f \), hence \( \varepsilon(D) : F \circ U(D) \rightarrow D \) is an isomorphism. It follows that \( K \) is fractioning. \( \square \)

Definition 4.2.6
Let \( K : \mathcal{E} \rightarrow \mathcal{F} \) and \( K' : \mathcal{E}' \rightarrow \mathcal{F}' \) be two fractioning propagators. A morphism from \( K \) to \( K' \) is a pair \((L, \overline{T})\) of propagators \( L : \mathcal{E} \rightarrow \mathcal{E}' \), \( \overline{T} : \mathcal{F} \rightarrow \mathcal{F}' \), such that \( \overline{T} \circ K = K' \circ L \).

Example 4.2.7
In section 4.1, the propagator \( P : \mathcal{E}_{\text{Fre}} \rightarrow \mathcal{E}_{\text{Cat}} \) is not fractioning, whereas the propagator \( K : \mathcal{E}_{\text{Comp}} \rightarrow \mathcal{E}_{\text{Cat}} \) is fractioning.

4.3 Filling propagators

Definition 4.3.1
A propagator \( J : \mathcal{E}_0 \rightarrow \mathcal{E} \) is filling if the freely generating functor \( F_J \) is full and faithful. Then the underlying functor \( U_J \) is the support functor with respect to \( J \).

From theorem 2.3.4, \( J \) is a filling propagator if and only if the unit \( \eta_J \) is a natural isomorphism:

\[ \eta_J : id_{\mathcal{R}(\mathcal{E}_0)} \cong U_J \circ F_J (= M_J). \]

Obviously, a conservative propagator is filling, the composite of filling propagators is filling, and a propagator which is equivalent to a filling one is also filling.

The next result gives a characterization of filling propagators in terms of their types, as defined in section 3.6. This result will not be used in this paper, and it is not proven either.

Theorem 4.3.2 (filling propagators)
A propagator \( J \) is filling if and only if the functor which underlies the morphism of projective types \( Ty(J) \) is full and faithful.

We now define a notion of distributor, which is a variant of the notion of distributor defined originally in [Bénabou, 1973].
Definition 4.3.3
In this paper, a **distributor** is a propagator $J : \mathcal{E}_0 \to \mathcal{E}$, which is an inclusion and which adds to $\mathcal{E}_0$:
- a copy of a projective sketch $\mathcal{E}_1$ which has no distinguished projective cone with empty base,
- some **transition** arrows from $\mathcal{E}_1$ to $\mathcal{E}_0$, i.e. some arrows with their source in $\mathcal{E}_1$ and their target in $\mathcal{E}_0$,
- some **transverse** commutative squares, i.e. some commutative squares $tr' \circ e_1 = e_0 \circ tr$, where $tr$ and $tr'$ are transition arrows, $e_1$ is in $\mathcal{E}_1$ and $e_0$ in $\mathcal{E}_0$,
- and some distinguished **transverse** projective cones, where a transverse projective cone has its vertex in $\mathcal{E}_1$, at least a point of its base in $\mathcal{E}_1$, and at least a point of its base in $\mathcal{E}_0$.

**Proposition 4.3.4**
A propagator which is equivalent to a distributor is filling.

**Proof.** Let $J : \mathcal{E}_0 \to \mathcal{E}$ be a filling propagator. For each realization $S$ of $\mathcal{E}_0$, the realization $F_J(S)$ of $\mathcal{E}$ is easy to compute: it coincides with $S$ on $\mathcal{E}_0$, and $F_J(S)(E) = \emptyset$ for all point $E$ not in $\mathcal{E}_0$. It follows immediately that $U_J \circ F_J(S) \cong S$, so that $F_J$ is full and faithful.

This proves that a distributor is a filling propagator, hence the proposition follows. □

In a distributor, the base of a transverse projective cone can be $E_1 \xrightarrow{tr} E_0 \xleftarrow{tr'} E_1$ for some transition arrow $tr$, so that it is possible to state that some transition arrows are monomorphic.

**Proposition 4.3.5**
Let $J$ be a distributor with at least one potential monomorphic transition arrow with source $E_1$ for each point $E_1$ of $\mathcal{E}_1$. Then the underlying functor $U_J : \text{Real}(\mathcal{E}) \to \text{Real}(\mathcal{E}_0)$ is faithful.

**Proof.** Let $\sigma_1, \sigma_2 : S \to S'$ be two morphisms of realizations of $\mathcal{E}$ such that $U(\sigma_1) = U(\sigma_2) : U(S) \to U(S')$. We have to prove that $\sigma_1(E) = \sigma_2(E)$ for all point $E$ of $\mathcal{E}$.

If $E = E_0$ is a point of $\mathcal{E}_0$, then $\sigma_i(E_0) = U(\sigma_i)(E_0)$ for $i = 1$ and 2, so that $\sigma_1(E_0) = \sigma_2(E_0)$.

Otherwise, $E = E_1$ is a point of $\mathcal{E}_1$, and there is a monomorphic transition arrow $tr : E_1 \to E_0$. For $i = 1$ and 2, from the naturality of $\sigma_i$ we get $S'(tr) \circ \sigma_i(E_1) = \sigma_i(E_0) \circ S(tr)$. Since $\sigma_1(E_0) = \sigma_2(E_0)$, we get $S'(tr) \circ \sigma_1(E_1) = S'(tr) \circ \sigma_2(E_1)$. Since $S'(tr)$ is a monomorphism, we get $\sigma_1(E_1) = \sigma_2(E_1)$. □

**Example 4.3.6**
In section 4.1, the propagator $P : \mathcal{E}_\mathfrak{gr} \to \mathcal{E}_\mathfrak{cat}$ is not filling, whereas the propagator $J : \mathcal{E}_\mathfrak{gr} \to \mathcal{E}_\mathfrak{comp}$ is filling. Indeed, it is equivalent to $J' : \mathcal{E}_\mathfrak{gr} \to \mathcal{E}_\mathfrak{comp}$ (see section 4.1), and it is easily checked that $J'$ is a distributor.

### 4.4 Decomposition of propagators

A propagator is, in general, neither fractioning nor filling. The following result proves that, up to equivalence, it can be decomposed as a filling propagator followed by a fractioning one. Actually, there are several ways to achieve such a decomposition. One systematic way stems from the proof which is given below.

**Theorem 4.4.1 (decomposition of propagators)**
Let $P : \mathcal{E}_0 \to \mathcal{E}$ be a propagator. There are a projective sketch $\mathcal{E}$, a fractioning propagator $K : \mathcal{E} \to \mathcal{E}$ and a filling propagator $J : \mathcal{E}_0 \to \mathcal{E}$ such that:

$$P \equiv K \circ J.$$  

In addition, it can be assumed that $J$ is a distributor.

**Proof.** Let $J : \mathcal{E}_0 \to \mathcal{E}$ be the distributor which adds to $\mathcal{E}_0$:
- a copy of the support $\mathcal{E}_1 = \text{Supp}(\mathcal{E})$ of $\mathcal{E}$ (so, $\mathcal{E}_1$ is a projective sketch without any distinguished projective cone),
- the transition arrows $tr_{E_1, E_0} : E_1 \to E_0$ for all points $E_1$ of $\mathcal{E}_1$ and $E_0$ of $\mathcal{E}_0$ such that $P(E_0) = E_1$. 

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the transverse commutative squares $tr_{E_1',E_0'} \circ e_1 = e_0 \circ tr_{E_1,E_0}$ for all arrows $e_1 : E_1 \to E_1'$ of $\mathcal{E}_t$ and $e_0 : E_0 \to E_0'$ of $\mathcal{E}_0$ such that $P(e_0) = e_1$. There is no distinguished transverse projective cone.

Now, let $\overline{\mathcal{E}}$ be made of $\mathcal{E}$ together with one identity for each point, so that the inclusion $\mathcal{E} \subseteq \overline{\mathcal{E}}$ is an equivalence. Let $K : \mathcal{E} \to \overline{\mathcal{E}}$ be the propagator such that:
- on $\mathcal{E}_0$, it coincides with $P$,
- on $\mathcal{E}_1$, it coincides with the inclusion $\text{Supp}(\mathcal{E}) \subseteq \mathcal{E} \subseteq \overline{\mathcal{E}}$,
- all transition arrow $tr_{E_1,E_0} : E_1 \to E_0$ is mapped to $id_{E_1} : E_1 \to E_1'$: indeed $K(E_0) = P(E_0) = E_0$ and $K(E_1) = E_1$.

Then all transverse commutative square $tr_{E_1',E_0'} \circ e_1 = e_0 \circ tr_{E_1,E_0}$ is preserved, since both $tr_{E_1',E_0'} \circ e_1$ and $e_0 \circ tr_{E_1,E_0}$ are mapped to $e_1$: indeed $K(e_0) = P(e_0) = e_1$ and $K(e_1) = e_1$.

Then obviously $P \equiv K \circ J$.

Finally, $K$ can be decomposed as $K = K_2 \circ K_1$, where $K_1$ maps the transition arrows to identities and $K_2$ is the distinction of the projective cones of $\overline{\mathcal{E}}$. From proposition 4.2.4 and theorem 4.2.3, both $K_1$ and $K_2$ are fractioning, so that $K$ itself is fractioning.

\[\begin{array}{c}
\mathcal{E} \\
\downarrow J \\
\mathcal{E}_0 \\
\end{array} \\
\begin{array}{c}
\mathcal{E} \\
\downarrow J \\
\mathcal{E}_0 \\
\end{array} \xrightarrow{K} \begin{array}{c}
\overline{\mathcal{E}} \\
\downarrow J \\
\mathcal{E}_0 \\
\end{array} \xrightarrow{\text{Real}(\mathcal{E})} \begin{array}{c}
\text{Real}(\overline{\mathcal{E}}) \\
\downarrow J \\
\text{Real}(\mathcal{E}_0) \\
\end{array} \xrightarrow{\text{Real}(\mathcal{E})} \begin{array}{c}
\text{Real}(\overline{\mathcal{E}}) \\
\downarrow J \\
\text{Real}(\mathcal{E}_0) \\
\end{array} \xrightarrow{F_{K}(f.f.)} \begin{array}{c}
\text{Real}(\overline{\mathcal{E}}) \\
\end{array} \xrightarrow{U_K(f.f.)} \begin{array}{c}
\text{Real}(\overline{\mathcal{E}}) \\
\end{array}
\]

\[\begin{array}{c}
E_1 \xrightarrow{e} E_1' \\
\downarrow tr \\
E_0 \end{array} \xrightarrow{K} \begin{array}{c}
E_1 \xrightarrow{id_{E_0}} E_1' \\
\downarrow tr_{E_1,E_0} \\
E_0 \end{array} \xrightarrow{J} \begin{array}{c}
E_0 \xrightarrow{id_{E_0}} E_0' \\
\end{array} \xrightarrow{J} \begin{array}{c}
E_0 \end{array}
\]

\[\begin{array}{c}
E_1 \xrightarrow{e} E_1' \\
\downarrow tr_{E_1,E_0} \\
E_0 \end{array} \xrightarrow{K} \begin{array}{c}
E_1 \xrightarrow{id_{E_0}} E_1' \\
\downarrow tr_{E_1,E_0} \\
E_0 \end{array} \xrightarrow{J} \begin{array}{c}
E_0 \xrightarrow{id_{E_0}} E_0' \\
\end{array} \xrightarrow{J} \begin{array}{c}
E_0 \end{array}
\]

\[\begin{array}{c}
E_1 \xrightarrow{e} E_1' \\
\downarrow tr_{E_1,E_0} \\
E_0 \end{array} \xrightarrow{K} \begin{array}{c}
E_1 \xrightarrow{id_{E_0}} E_1' \\
\downarrow tr_{E_1,E_0} \\
E_0 \end{array} \xrightarrow{J} \begin{array}{c}
E_0 \xrightarrow{id_{E_0}} E_0' \\
\end{array} \xrightarrow{J} \begin{array}{c}
E_0 \end{array}
\]

**Example 4.4.2**

In section 4.1, the propagator $P : \mathcal{E}_{\mathcal{F}_r} \to \mathcal{E}_{\mathcal{C}_at}$ has been decomposed as $P \equiv K' \circ J$ with $J : \mathcal{E}_{\mathcal{F}_r} \to \mathcal{E}_{\mathcal{C}_comp}$ filling and $K' : \mathcal{E}_{\mathcal{C}_comp} \to \mathcal{E}_{\mathcal{C}_at}$ fractioning. This decomposition of $P$ corresponds to the last variant above: both operations $\text{comp}$ and $\text{sedl}$, which do not occur in $\mathcal{E}_{\mathcal{F}_r}$, are introduced as partial operations in $\mathcal{E}_{\mathcal{C}_comp}$, then they are made total in $\mathcal{E}_{\mathcal{C}_at}$.
5 Diagrammatic specifications

In this section we define some basic notions related to logic, like syntactic entailment and semantic consequence, in the general framework of fractioning propagators.

5.1 Specifications and domains

Definition 5.1.1
Let $P : \mathcal{E} \to \mathcal{E}$ be a propagator. The category of (diagrammatic) specifications with respect to $P$, or $P$-specifications, is the category of realizations of $\mathcal{E}_0$, and the category of (diagrammatic) domains with respect to $P$, or $P$-domains, is the category of realizations of $\mathcal{E}$.

$$\text{Spec}(P) = \text{Real}(\mathcal{E}_0) \quad \text{and} \quad \text{Dom}(P) = \text{Real}(\mathcal{E}).$$

Of course, this definition can be used when the propagator is fractioning. On the other hand, from the decomposition theorem 4.4.1, all propagator $P : \mathcal{E}_0 \to \mathcal{E}$ can be decomposed as $P = K \circ J$, with $K : \mathcal{E}_0 \to \mathcal{E}$ fractioning and $J : \mathcal{E}_0 \to \mathcal{E}$ filling. Then, the $K$-domains are the $P$-domains, and each $P$-specification $S_0$ freely generates a $K$-specification $S = F_J(S_0)$, which is such that $F_P(S_0) = F_K(S)$. In addition, from the proof of theorem 4.4.1, $J$ can be chosen in such a way that $S$ is essentially the same as $S_0$.

Hence, from now on, $K : \mathcal{E} \to \mathcal{E}$ is a fractioning propagator.

Definition 5.1.2
A $K$-specification $S$ is saturated if the morphism $\eta_{K,S} : S \to M_K(S)$ is an isomorphism.

Proposition 5.1.3 (saturated specifications)
Let $S$ be a $K$-specification. Then the $K$-specification $M_K(S)$ is saturated.

Proof. Since $\eta \circ M$ is a natural isomorphism, the morphism $\eta \circ M(S) = \eta_{M(S)} : M(S) \to M(M(S))$ is an isomorphism. $\square$

5.2 Syntactic entailment

Definition 5.2.1
A morphism $\sigma : S \to S'$ of $K$-specifications is a syntactic entailment if the morphism of $K$-domains $F_K(\sigma)$ is an isomorphism:

$$S \xrightarrow{\sigma} S' \quad \text{if and only if} \quad F_K(\sigma) : F_K(S) \xrightarrow{\cong} F_K(S').$$

Proposition 5.2.2
Let $\sigma : S \to S'$ be a morphism of $K$-specifications. Then $S \xrightarrow{\sigma} S'$ if and only if $M_K(\sigma)$ is an isomorphism.

Proof. If $F(\sigma)$ is an isomorphism, then clearly $M(\sigma)$ is an isomorphism, because $M = U \circ F$ (and this is true for any propagator $K$).

On the other hand, since $K$ is fractioning, the functor $U$ is full and faithful. So, if a morphism $\delta : D \to D'$ is such that $U(\delta)$ is an isomorphism, then $\delta$ itself is an isomorphism. This can be applied to $\delta = F(\sigma)$: if $M(\sigma)$ is an isomorphism, then $F(\sigma)$ is an isomorphism. $\square$

Theorem 5.2.3
Let $\sigma : S \to S'$ be a morphism of $K$-specifications. Then $S \xrightarrow{\sigma} S'$ if and only if there is a morphism of $K$-specifications $\alpha : S' \to M_K(S)$ such that $\alpha \circ \sigma = \eta_{K,S}$ and $M_K(\sigma) \circ \alpha = \eta_{K,S'}$. In such a case, $\alpha = (M_K(\sigma))^{-1} \circ \eta_{S'}$. 

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The condition in the theorem means that the commutative square $\eta_{K,S'} \circ \sigma = M_K(\sigma) \circ \eta_{K,S}$ is split:

\[
\begin{array}{ccc}
S & \xrightarrow{\eta_S} & M(S) \\
\sigma \downarrow & & \downarrow \circ \alpha \\
S' & \xrightarrow{\eta_{S'}} & M(S')
\end{array}
\]

**Proof.** Let $S \xrightarrow{\sigma} S'$, so that $M(\sigma)$ is an isomorphism, by proposition 5.2.2. Let $\alpha = (M(\sigma))^{-1} \circ \eta_{S'} : S' \to M(S)$. Then $\alpha \circ \sigma = (M(\sigma))^{-1} \circ \eta_{S'} \circ \sigma = (M(\sigma))^{-1} \circ M(\sigma) \circ \eta_{S} = \eta_{S}$, and $M(\sigma) \circ \alpha = M(\sigma) \circ (M(\sigma))^{-1} \circ \eta_{S'} = \eta_{S'}$.

Now, let $\alpha : S' \to M_K(S)$ be such that $\alpha \circ \sigma = \eta_{K,S}$ and $M_K(\sigma) \circ \alpha = \eta_{K,S'}$. Let us prove that $\mu_S \circ M(\alpha) : M(S') \to M(S)$ is an inverse of $M(\sigma)$. Since $K$ is fractioning, the monad $M$ is idempotent, which means that $\mu$ is a natural isomorphism, with inverse $M \circ \eta$. (see corollary 2.3.5).

On one hand, from $\alpha \circ \sigma = \eta_{S}$, we get:

\[\mu_S \circ M(\alpha) \circ M(\sigma) = \mu_S \circ M(\alpha \circ \sigma) = \mu_S \circ M(\eta_{S}) = id_{M(S)}\]

On the other hand, from $M(\sigma) \circ \alpha = \eta_{S'}$, we get $M^{2}(\sigma) \circ M(\alpha) = M(\eta_{S'})$, so that (thanks to the naturality of $\mu$):

\[M(\sigma) \circ \mu_S \circ M(\alpha) = M_{S'} \circ M^{2}(\sigma) \circ M(\alpha) = \mu_{S'} \circ M(\eta_{S'}) = id_{M(S')}\]

So, $M(\sigma)$ is an isomorphism, with inverse $\mu_S \circ M(\alpha)$. □

### 5.3 Syntactic deduction steps

**Definition 5.3.1**

A deduction rule with respect to $K$ is an arrow $r : H \to C$ in $\mathcal{E}$. The point $H$ is the hypothesis and the point $C$ is the conclusion of the rule $r$.

A deduction rule $r : H \to C$ can be written as $r : \mathcal{C}_H \to \mathcal{C}_C$, or simply as $\mathcal{C}_H \to \mathcal{C}_C$. From theorem 4.2.2, up to equivalence of sketches, the hypothesis and conclusion of a rule are points of $\mathcal{E}$, and there are two kinds of rules:

- a deduction rule $r : H \to C$ is passive if $r$ is an arrow of $\mathcal{E}$,
- a deduction rule $r : H \to C$ is active if $r$ is the inverse of an arrow $e : C \to H$ of $\mathcal{E}$.

Deduction rules can be composed, as arrows in $\mathcal{E}$.

The Yoneda contravariant realization $Y_\mathcal{E}$ of $\mathcal{E}$ yields illustrations for active deduction rules. Indeed, let $e : C \to H$ be an arrow of $\mathcal{E}$, and let $r = e^{-1} : H \to C$ be the corresponding active deduction rule. The image of $e : C \to H$ by $Y_\mathcal{E}$ is a morphism of realizations of $\mathcal{E}$:

\[Y_\mathcal{E}(e) : Y_\mathcal{E}(H) \to Y_\mathcal{E}(C)\]

Since the Yoneda realization is contravariant, the source and target of the morphism $Y_\mathcal{E}(e)$ are respectively (the images of) the hypothesis and the conclusion of the rule $r$. The morphism $Y_\mathcal{E}(e)$ becomes an isomorphism in $S(\mathcal{E})$. In this way, $Y_\mathcal{E}(e)$ illustrates the deduction rule $r : H \to C$. For instance, this is the way the definition of categories is illustrated by functors of composite graphs in section 2.2.

Now, let $r = e^{-1} : H \to C$ be an active deduction rule.

Let $S$ be a $K$-specification and $x \in S(H)$. The inverse image of $x$ by $S(e)$ can be any subset $(S(e))^{-1}(x)$ of $S(C)$:

\[(S(e))^{-1}(x) \subseteq S(C)\]

When $S$ is saturated, $(S(e))^{-1}(x)$ is made of exactly one element $y$ of $S(C)$.

We now define the “simplest” morphism $\sigma : S \to S'$ with source $S$ such that, if $x' = \sigma(H)(x)$, the inverse image of $x'$ by $S'(e)$ is made of exactly one element $y'$ of $S'(C)$:

\[(S'(e))^{-1}(x') = \{y'\}\]
For this purpose, let $\Phi : \mathcal{E} \rightarrow \mathcal{E}_1$ be the inclusion propagator which adds points $H_1$ and $C_1$, arrows $h : H_1 \rightarrow H$, $c : C_1 \rightarrow C$ and $e_1 : C_1 \rightarrow H_1$, and two distinguished projective cones: the first one with vertex $H_1$ and empty base, the second one with vertex $C_1$, base $C \xrightarrow{e} H \xleftarrow{h} H_1$ and projections $c$ and $e_1$.

The set-valued realizations of $\mathcal{E}_1$ are, up to isomorphisms, the pairs $S_1 = (S, x)$ where $S$ is a set-valued realization of $\mathcal{E}$ and $x$ is an element of $S(H)$. Then clearly $S = U_\Phi(S_1)$.

When $\mathcal{E}$ contains only $C \xrightarrow{e} H$, then $\mathcal{E}_1$ is as follows:

$$
\begin{align*}
\mathcal{E}_1 : & \quad \begin{array}{c}
C_1 \xrightarrow{c} H_1 \\
\downarrow e \\
C \xrightarrow{e} H
\end{array} & \text{with 2 dpcs:} & \begin{array}{c}
H_1 \\
\downarrow e_1 \\
H
\end{array} & \text{(empty base)} & \begin{array}{c}
C \xrightarrow{c} H \xleftarrow{h} H_1 \\
\downarrow e_1
\end{array}
\end{align*}
$$

Let $\overline{\Phi} : \overline{\mathcal{E}} \rightarrow \overline{\mathcal{E}_1}$ be obtained by a similar construction from $\overline{\mathcal{E}}$. Then the inclusion $K_1 : \mathcal{E}_1 \rightarrow \overline{\mathcal{E}_1}$ is a fractioning propagator and the following square is a pushout:

$$
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{K} & \overline{\mathcal{E}} \\
\Phi \downarrow & & \downarrow \Phi \overline{\mathcal{E}_1} \\
\mathcal{E}_1 & \xrightarrow{K_1} & \overline{\mathcal{E}_1}
\end{array}
$$

Let $S$ be a set-valued realization of $\mathcal{E}$ and $x \in S(H)$. Let $\overline{x}$ denote the image of $x$ in $M_K(S)(H)$ by the map $\eta_K : S(H) \rightarrow M_K(S)(H)$. Since $K$ is an inclusion, $\overline{x} \in F_K(S)(H)$. The next result is easy to prove.

**Lemma 5.3.2**

*With the above notations, if $S_1 = (S, x)$ then $F_{K_1}(S_1) \cong (F_K(S), \overline{x})$, naturally in $S_1$ in $\text{Real}(\mathcal{E}_1)$.*

Now, let $\Psi : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ denote the fractioning propagator which adds an inverse $r_1 : H_1 \rightarrow C_1$ to $e_1$.

When $\mathcal{E}$ contains only $C \xrightarrow{e} H$, then $\mathcal{E}_2$ is as follows:

$$
\begin{align*}
\mathcal{E}_2 : & \quad \begin{array}{c}
C_1 \xrightarrow{c} H_1 \\
\downarrow e_1 \\
C \xrightarrow{e} H
\end{array} & \text{with 2 dpcs} & \begin{array}{c}
r_1 \circ e_1 = \text{id}_{C_1} \\
e_1 \circ r_1 = \text{id}_{H_1}
\end{array} & \text{and with 2 composites:} & \begin{array}{c}
\text{(like $\mathcal{E}_1$)}
\end{array}
\end{align*}
$$

Let $\overline{\Psi} : \overline{\mathcal{E}_1} \rightarrow \overline{\mathcal{E}_2}$ be obtained by a similar construction from $\overline{\mathcal{E}_1}$. Then the inclusion $K_2 : \mathcal{E}_2 \rightarrow \overline{\mathcal{E}_2}$ is a fractioning propagator and the following square is a pushout:

$$
\begin{array}{ccc}
\mathcal{E}_1 & \xrightarrow{K_1} & \overline{\mathcal{E}_1} \\
\Phi \downarrow & & \downarrow \Phi \overline{\mathcal{E}_2} \\
\mathcal{E}_2 & \xrightarrow{K_2} & \overline{\mathcal{E}_2}
\end{array}
$$

Let $S_1 = (S, x)$ be a $K_1$-specification, so that $S = U_\Phi(S_1)$, and let $S'_1 = M_{K_1}(S_1)$ and $S' = U_\Phi(S'_1)$. Then, from $\eta_{K_1,s_1} : S_1 \rightarrow S'_1$, we get a morphism of $K$-specifications:

$$
\sigma = U_\Phi(\eta_{K_1,s_1}) : S \rightarrow S'.
$$

Let $x' = \sigma(H)(x) \in S'(H)$, then the inverse image of $x'$ by $S'(e)$ is made of one point: namely, $(S'(e))^{-1}(x') = \{y'\}$ where $y' = S'((r_1)(x')) \in S'(C)$. On the other hand, if $(S(e))^{-1}(x)$ is made of one point, then $\sigma = \text{id}_S : S \rightarrow S$.
Definition 5.3.3
Let $r : H \rightarrow C$ be a deduction rule with respect to $K$, and let $x$ be an element in $S(H)$. The deduction step with respect to $K$ associated to $r$ and $x$ is a morphism of $K$-specifications with source $S$. If $r$ is a passive deduction rule, it is the identity morphism $id_S : S \rightarrow S$. If $r=e^{-1}$ is an active deduction rule, it is the morphism $\sigma : S \rightarrow S'$ as defined above.

Proposition 5.3.4 (syntactic deduction step)
Let $\sigma : S \rightarrow S'$ be a deduction step. Then it is a syntactic entailment: $S \vdash \sigma \rightarrow S'$.

Proof. Let $\sigma : S \rightarrow S'$ be the deduction step associated to the rule $r : H \rightarrow C$ and to $x \in S(H)$. Let us prove that $F_K(\sigma) : F_K(S) \rightarrow F_K(S')$ is an isomorphism.

If $r$ is passive then $\sigma$ is the identity, so that $F_K(\sigma)$ is the identity.

If $r$ is active, lemma 5.3.2 proves that $F_K(\sigma) \cong U_\Phi(F_K(\sigma))$. Since $\Phi$ is conservative, it follows that, up to isomorphism, $F_K(\sigma)$ is the deduction step associated to the rule $r : H \rightarrow C$ and to $\bar{x} \in F_K(S)(H)$. Since $F_K(S)$ is saturated, this deduction step is the identity. □

It follows that any finite composition of deduction steps is a syntactic entailment.

In the opposite direction, it can be proven that all syntactic entailment (for instance $\eta_K : S \rightarrow M_K(S)$) can be obtained from syntactic deduction steps by nested inductive limits of various types.

5.4 Models

Definition 5.4.1
Let $D$ be a $K$-domain. The contravariant functor of models $\text{Mod}_K(-, D) : \text{Spec}(K) \rightarrow \text{Set}$ is:

$$\text{Mod}_K(-, D) = \text{Hom}_{\text{Real}(\Phi)}(F_K(-), D).$$

It follows from the generated realization theorem 3.4.1 that:

$$\text{Mod}_K(-, D) \cong \text{Hom}_{\text{Real}(\mathcal{E})}(-, U_K(D)).$$

So, for each $K$-specification $S$, the models of $S$ with values in $D$ are the morphisms from $F_K(S)$ to $D$ in $\text{Real}(\Phi)$, and they can be identified with the morphisms from $S$ to $U_K(D)$ in $\text{Real}(\mathcal{E})$:

$$\text{Mod}_K(S, D) = \text{Hom}_{\text{Real}(\Phi)}(F_K(S), D) \cong \text{Hom}_{\text{Real}(\mathcal{E})}(S, U_K(D)).$$

From the definition of morphisms in section 3.3, a model $\omega$ of $S$ with values in $D$ can be identified with a natural transformation between the functors underlying $S$ and $U_K(D)$: it is made of a map $\omega_E : S(E) \rightarrow D(K(E))$ for each point $E$ of $\mathcal{E}$, naturally in $E$.

In this paper, a model $\omega$ of $S$ with values in $D$ is illustrated as $\omega : S \rightarrow D$, so that (Kleisli categories could be invoked here, see [Mac Lane, 1971, p.143]):

$$(\omega : S \rightarrow D) = (\omega : F_K(S) \rightarrow D).$$

Let $\sigma : S \rightarrow S'$ be a morphism of $K$-specifications. The map $\text{Mod}_K(\sigma, D) : \text{Mod}_K(S', D) \rightarrow \text{Mod}_K(S, D)$ maps $\omega' : S' \rightarrow D$, i.e. $\omega' : F_K(S') \rightarrow D$, to $\omega' \circ F_K(\sigma) : F_K(S) \rightarrow D$, which is denoted, in this paper, $\omega' \circ \sigma : S \rightarrow D$, so that:

$$(\omega' \circ \sigma : S \rightarrow D) = (\omega' \circ F_K(\sigma) : F_K(S) \rightarrow D).$$

Proposition 5.4.2
The map $\text{Mod}_K(\eta_K, D)$ is a bijection: $\text{Mod}_K(M_K(S), D) \cong \text{Mod}_K(S, D)$.
So, the models of a $K$-specification $S$ can be identified with the models of $M_K(S)$.

**Proof.** From theorem 2.3.4, since $U$ is full and faithful, $F \circ \eta$ is a natural isomorphism $F \circ \eta : F \cong F \circ U \circ F$. So, the map $\text{Hom}_{R_M}(F_K(\eta_S), D)$, i.e. the map $\text{Mod}_K(\eta_S, D)$, is a bijection. □

In such a general setting, there is no canonical notion of morphism of models, hence no category of models of $S$ with values in $D$. However, in many important special cases, there is such a category of models; then the contravariant functor of models is:

$$\text{Mod}_K(-, D) : \text{Spec}(K) \to \text{Cat}.$$

### 5.5 Semantic consequence

**Definition 5.5.1**

Let $D$ be a $K$-domain. A morphism $\sigma : S \to S'$ of $K$-specifications is a semantic consequence with respect to $D$ if the map $\text{Mod}(\sigma, D)$ is a bijection:

$$S \xrightarrow{\sigma} D S' \quad \text{if and only if} \quad \text{Mod}(\sigma, D) : \text{Mod}(S', D) \cong \text{Mod}(S, D),$$

### 5.6 Soundness

Entailment and consequence are related by the soundness property, which is easily derived from the properties of adjunction.

**Theorem 5.6.1 (soundness)**

Let $D$ be a $K$-domain. For all morphism of $K$-specifications $\sigma : S \to S'$, if $\sigma$ is a syntactic entailment, then it is a semantic consequence with respect to $D$:

$$\text{if} \quad S \xrightarrow{\sigma} D S' \quad \text{then} \quad S \xrightarrow{\sigma} D S'.$$

This means that all fractioning propagators are sound.

**Proof.** Let $\sigma : S \to S'$ be a morphism of $K$-specifications. Since $\eta$ is a natural transformation, we have $\eta_S \circ \sigma = M_\sigma \circ \eta_S$. Hence $\text{Mod}(\sigma, D) \circ \text{Mod}(\eta_S, D) = \text{Mod}(\eta_S, D) \circ \text{Mod}(M_\sigma, D)$. On one hand, since $\sigma$ is a conservative morphism, $M_\sigma$ is an isomorphism, and $\text{Mod}(M_\sigma, D)$ is a bijection. On the other hand, from proposition 5.4.2, both $\text{Mod}(\eta_S, D)$ and $\text{Mod}(\eta_S, D)$ are bijections. It follows that $\text{Mod}(\sigma, D)$ is also a bijection. □

**Proposition 5.6.2**

Let $\sigma : S \to S'$ be a morphism of $K$-specifications. If $\sigma$ is a semantic consequence with respect to all $K$-domain $D$, then it is a syntactic entailment:

$$\text{if} \quad S \xrightarrow{\sigma} D S' \text{ for all } D \quad \text{then} \quad S \xrightarrow{\sigma} S'.$$

**Proof.** The assumption means that the map $\text{Mod}(\sigma, D) : \text{Mod}(S', D) \to \text{Mod}(S, D)$ is a bijection for all domain $D$. From the definition of models, this means that the map $\text{Hom}(F(\sigma), D) : \text{Hom}(F(S'), D) \to \text{Hom}(F(S), D)$ is a bijection for all domain $D$.

So, when $D = F(S)$, the map $\delta \mapsto \delta \circ F(\sigma)$ is a bijection $\text{Hom}(F(\sigma), F(S)) : \text{Hom}(F(S'), F(S)) \cong \text{Hom}(F(S), F(S));$ hence, there is a unique morphism $\tau : F(S') \to F(S)$ such that $\tau \circ F(\sigma) = \text{id}_{F(S)}$.

Now, when $D = F(S')$, the map $\delta \mapsto \delta \circ F(\sigma)$ is a bijection $\text{Hom}(F(\sigma), F(S')) : \text{Hom}(F(S'), F(S')) \cong \text{Hom}(F(S), F(S'))$. This map is such that $F(\sigma) \circ \tau \mapsto F(\sigma) \circ \tau \circ F(\sigma)$, which is equal to $F(\sigma)$, since $\tau \circ F(\sigma) = \text{id}_{F(S')}$. But clearly $\text{id}_{F(S')} \mapsto F(\sigma)$, so that $F(\sigma) \circ \tau = \text{id}_{F(S')}$. So, $F(\sigma)$ is an isomorphism, with inverse $\tau$. □
5.7 Satisfaction

Here it is proven that the relation of semantic consequence, between two specifications, can also be obtained from a relation of satisfaction, between a model and a specification. The satisfaction only makes sense when there is some notion of signature of a specification. In our context, the signatures are the set-valued realizations of a projective sketch $ς_0$, such that there is a propagator $J : ς_0 → ⃗ς$.

More precisely, let $K : ς_0 → ⃗ς$ be a fractioning propagator, together with a homomorphism $\overline{J} = (J, \overline{J}) : K_0 → K$.

$$
\begin{array}{c}
\text{ς} \\
\downarrow \quad K
\end{array}
\begin{array}{c}
\text{Greg} \\
\downarrow \quad \overline{J}
\end{array}
\begin{array}{c}
\text{ς}
\end{array}
$$

Let $S_0$ be a $K_0$-specification and $D_0$ a $K_0$-domain.

For all $K$-specification $S$ such that $U_J(S) = S_0$ and all $K$-domain $D$ such that $U_J(D) = D_0$, the underlying functor $U_J : \text{Real}(ς) → \text{Real}(ς_0)$ determines a map:

$$(U_J)_*: \text{Mor}_{\text{Real}(ς)}(S, U_K(D)) → \text{Mor}_{\text{Real}(ς_0)}(S_0, U_J(U_K(D))),$$

and since $U_J ∘ U_K = U_{K_0} ∘ U_J$ this map is:

$$(U_J)_*: \text{Mor}_{\text{Real}(ς)}(S, U_K(D)) → \text{Hom}_{\text{Real}(ς_0)}(S_0, U_{K_0}(D_0)).$$

So, by adjunction, we get a map:

$$(U_J)_*: \text{Mod}_K(S, D) → \text{Mod}_{K_0}(S_0, D_0),$$

such that:

$$\omega \mapsto (U_J(\omega_0))^*.$$ 

This map is natural both in $S$ and in $D$.

**Definition 5.7.1** For all $K$-specification $S$ such that $U_J(S) = S_0$ and all $K$-domain $D$ such that $U_J(D) = D_0$, the underlying model map with respect to $\overline{J}$ is the map $\omega \mapsto (U_J(\omega_0))^*$:

$$(U_J)_*: \text{Mod}_K(S, D) → \text{Mod}_{K_0}(S_0, D_0).$$

**Definition 5.7.2** Let $S$ be a $K$-specification such that $U_J(S) = S_0$, and $D$ a $K$-domain such that $U_J(D) = D_0$. A model $\omega_0$ of $S_0$ with values in $D_0$ satisfies $S$ with respect to $D$ if $\omega_0$ is in the image of $\text{Mod}(S, D)$ by $(U_J)_*$. This is denoted:

$$\omega_0 \models_{D} S.$$ 

Let us now make the following assumption $(IN J)$:

The map $(U_J)_*$ is injective, for all $K$-specification $S$ and all $K$-domain $D$.

This happens when $J$ is a filling propagator which satisfies the condition of proposition 4.3.5.

Under this assumption $(IN J)$, the map $(U_J)_*$ can be used for identifying $\text{Mod}(S, D)$ and its image in $\text{Mod}(U_J(S), U_J(D))$. Then, we can say that $\omega_0 : S_0 → D_0$ satisfies $S$ with respect to $D$ if and only if it “is” a model of $S$ with values in $D$.

**Theorem 5.7.3**

Under assumption $(IN J)$, let $\sigma : S → S'$ be a morphism of $K$-specifications such that $U_J(S) = U_J(S') = S_0$ and $U_J(\sigma) = id_{S_0}$, and let $D$ be a $K$-domain such that $U_J(D) = D_0$.

Then $S \models_{D} S'$ if and only if, for all $\omega_0 : S_0 → D_0$, if $\omega_0 \models_{D} S$ then $\omega_0 \models_{D} S'$.
Proof. Because of the naturality of the map \((U_f)_S^D\) with respect to \(S\), the following triangle \(T\) is commutative:

\[
\begin{array}{ccc}
\text{Mod}(S, D) & \xrightarrow{(U_f)_S^D} & \text{Mod}(U_f(S), U_f(D)) \\
\text{Mod}(\sigma, D) \downarrow & & \downarrow \\
\text{Mod}(S', D) & \xrightarrow{(U_f)_{S'}^D} & \text{Mod}(U_f(S'), U_f(D))
\end{array}
\]

Let us assume that \(S \xrightarrow{\sigma} S'\), i.e. that \(\text{Mod}(\sigma, D)\) is bijective. Let \(\omega_0 : S_0 \rightarrow D_0\) be such that \(\omega_0 \mapsto D\), which means that \(\omega_0\) is in the image of the map \((U_f)_S^D\). Then, since \(\text{Mod}(\sigma, D)\) is surjective, \(\omega_0\) is in the image of the map \((U_f)_S^D \circ \text{Mod}(\sigma, D)\), which is equal to \((U_f)_{S'}^D\) because of the commutativity of \(T\), so that \(\omega_0 \mapsto \tau_{\sigma} S'\).

On the other hand, the commutativity of \(T\) together with the assumption \((INJ)\) proves that the map \(\text{Mod}(\sigma, D)\) is injective. Now, let us assume that for all \(\omega_0 : S_0 \rightarrow D_0\), if \(\omega_0 \mapsto D\) then \(\omega_0 \mapsto D\). For all \(\omega : S \rightarrow D\), let \(\omega_0 = (U_f)_S^D(\omega)\), so that \(\omega_0 \mapsto D\). Then \(\omega_0 \mapsto D\) such that \(\omega' : S' \rightarrow D\), i.e. such that \(\omega_0 = (U_f)_{S'}^D(\omega')\). Because of the assumption \((INJ)\), this \(\omega'\) is uniquely determined. So, we get a map \(f : \text{Mod}(S, D) \rightarrow \text{Mod}(S', D)\), and because of the assumption \((INJ)\), \(f\) is an inverse to \(\text{Mod}(\sigma, D)\). It follows that \(\text{Mod}(\sigma, D)\) is bijective, so that \(S \xrightarrow{\tau_{\sigma}} S'\). \(\Box\)

6 About logic

In this last section, we outline some basic links between our diagrammatic specification techniques and issues in logic. First, we look at equational diagrammatic specifications, and then, more generally, at institutions.

6.1 About equational logic

In the context of algebraic specifications, as for instance in [Goguen, Thatcher and Wagner, 1976], an equational specification is defined in three steps: first a set of sorts, then a signature (i.e. a structured set of operators) on this set of sorts, and finally a set of equations on this signature. Some strings of sorts are used for introducing the operators, and some terms (composed from operators) are used for introducing the equations.

For example, an equational specification of naturals \(S_{nat}\) can be defined as follows:

- sorts: \(N\),
- operators: \(s : N \rightarrow N\), \(a : N \rightarrow N\), \(\lambda : N \rightarrow N\), with the strings of sorts \(NN\) and \(\lambda\) (empty string),
- equations: \(a(x, y) = x\) and \(a(x, s(y)) = s(a(x, y))\) where \(x\) and \(y\) are variables of sort \(N\). These equations can be written without variables, as relations between composed arrows. For instance, the second equation can be written as \(a \circ \text{fact(id}_N) = s \circ a : NN \rightarrow N\), with one identity arrow \(\text{id}_N : N \rightarrow N\), one factorization arrow \(\text{fact(id}_N, s) : N \rightarrow NN\) and two composed arrows.

The construction of an equational specification makes use of three successive propagators: \(P_s\) for sorts, \(P_o\) for operators, and \(P_e\) for equations.

Sorts.

The propagator \(P_s : E_{0,s} \rightarrow E_s\) is the usual one from a projective sketch of sets to projective sketch of monoids.

The projective sketch \(E_{0,s}\) is the simplest sketch of sets: it is made of one point \(\text{Sort}\) (similar to \(P_f\)). So, a \(P_s\)-specification \(S_s\) is a set of sorts.

The projective sketch \(E_s\) is a sketch of monoids: it contains the points \(\text{Sort}^0\), \(\text{Sort}\), \(\text{Sort}^2\), two arrows \(p_1, p_2 : \text{Sort}^2 \rightarrow \text{Sort}\) and two dpc’s: one with vertex \(\text{Sort}^0\) and empty base, another one with vertex \(\text{Sort}^2\), with two-points base \(\{\text{Sort}, \text{Sort}\}\) (discrete, i.e. without any arrow) and projections \(p_1, p_2\). The
point \( \textit{Sort} \) will be interpreted as the set of strings of sorts, \( \textit{Sort}^0 \) as a one-element set, and \( \textit{Sort}^2 \) as the set of pairs of strings of sorts. In \( \mathcal{S} \), two more arrows \( \lambda : \textit{Sort}^0 \to \textit{Sort} \) and \( \kappa : \textit{Sort}^2 \to \textit{Sort} \) stand respectively for the empty string of sorts and the concatenation of strings of sorts. There are additional features in \( \mathcal{S} \), in order to ensure that \( \kappa \) will be interpreted as an associative operation and \( \lambda \) as its unit.

So, the functor \( F_{\textit{P}} \) freely generates the strings of sorts.

The propagator \( P_\alpha \) is decomposed, according to theorem 4.4.1, as \( P_\alpha = K_\alpha \circ J_\alpha \), with an intermediate projective sketch \( \mathcal{E}_\alpha \) of partial monoids.

**Operators.**

The propagator \( P_\alpha : \mathcal{E}_{0,\alpha} \to \mathcal{S}_\alpha \) is similar to the propagator which has been considered in the previous sections, from a projective sketch of directed graphs to a projective sketch of categories. There is a point \( \textit{Op} \) (similar to \( \textit{Ar} \)) which stands for the set of operators in \( \mathcal{E}_{0,\alpha} \) and for the set of terms in \( \mathcal{S}_\alpha \). However, because of arities, \( P_\alpha \) is somewhat larger than that.

The sketch \( \mathcal{E}_{0,\alpha} \) contains \( \mathcal{E}_\alpha \), not only \( \mathcal{E}_{0,\alpha} \), in order to allow the definition of multivariate operators and constant operators. So, a \( P_\alpha \)-specification \( S_\alpha \) is a signature, in the equational meaning.

The inclusion propagator \( J_{\alpha,\alpha} : \mathcal{E}_\alpha \to \mathcal{E}_{0,\alpha} \) is filling. Let \( S_\alpha \) be a \( P_\alpha \)-specification. Then \( S_\alpha \) is a \( S_\alpha \)-sorted signature if \( U_{J_{\alpha,\alpha}}(S_\alpha) \) can be deduced from \( S_\alpha \), more precisely if \( F_{J_{\alpha,\alpha}}(S_\alpha) \Rightarrow U_{J_{\alpha,\alpha}}(S_\alpha) \), as \( K_\alpha \)-specifications.

The sketch \( \mathcal{E}_\alpha \), besides identity and composed arrows, also takes care of projection and factorization arrows. So, the functor \( F_{\textit{P}} \) freely generates the terms, in their categorical version, i.e. without variables.

The propagator \( P_\alpha \) is decomposed, according to theorem 4.4.1, as \( P_\alpha = K_\alpha \circ J_\alpha \), with an intermediate projective sketch \( \mathcal{E}_\alpha \) which contains the sketch of projective graphs.

**Equations.**

The propagator \( P_\alpha : \mathcal{E}_{0,\alpha} \to \mathcal{S}_\alpha \) is the propagator for equational specifications.

The sketch \( \mathcal{E}_{0,\alpha} \) contains \( \mathcal{E}_\alpha \) and a point \( \textit{Eq} \) for equations, with a potential monomorphism from \( \textit{Eq} \) to a point \( \textit{St} \) which stands (thanks to a dpc) for the set of pairs of terms with the same source and target.

So, a \( P_\alpha \)-specification \( S_\alpha \) is an equational specification.

The inclusion propagator \( J_{\alpha,\alpha} : \mathcal{E}_\alpha \to \mathcal{E}_{0,\alpha} \) is filling. Let \( S_\alpha \) be a \( P_\alpha \)-specification. Then the signature of \( S_\alpha \) is \( S_\alpha \) if \( U_{J_{\alpha,\alpha}}(S_\alpha) \) can be deduced from \( S_\alpha \), more precisely if \( F_{J_{\alpha,\alpha}}(S_\alpha) \Rightarrow U_{J_{\alpha,\alpha}}(S_\alpha) \), as \( K_\alpha \)-specifications.

The sketch \( \mathcal{E}_\alpha \) adds deduction rules, in such a way that the interpretation of \( \textit{Eq} \) in a realization of \( \mathcal{E}_\alpha \) is a congruence, i.e. an equivalence relation which is compatible with the composition of terms. So, the functor \( F_{\textit{P}} \) freely generates the congruence from the equations, i.e. the theorems from the axioms.

The propagator \( P_\alpha \) is decomposed, according to theorem 4.4.1, as \( P_\alpha = K_\alpha \circ J_\alpha \), with an intermediate projective sketch \( \mathcal{E}_\alpha \) for derived equational specifications.

To sum up, the definition of equational specifications makes use of the following commutative diagram of projective sketches and propagators:

![Diagram of projective sketches and propagators](image)

The domain of values is the realization \( D_{\textit{set}} \) of \( \mathcal{E}_\alpha \) which interprets the sorts as sets, the operations as maps, and the equations as identities between maps.
6.2 About institutions

The theory of institutions [Goguen and Burstall, 1992] defines some notions of logic in a very general setting. Diagrammatic specifications are quite different: they are restricted to projectively sketchable structures, in order to gain some effectiveness; and they do not assume any notion of formula or sentence, because diagrammatic specifications have been made for applications to computer languages which do not involve such notions. However, diagrammatic specifications can easily be related to institutions, more precisely to chartered institutions.

The idea is to consider a fractioning propagator $K_0 : \mathcal{E}_0 \to \overline{\mathcal{E}_0}$, together with a point $Sen$ in $\mathcal{E}_0$ and with a $K_0$-domain $D_0$, such that the interpretation of the point $K_0(\text{Sen})$ by $D_0$ is the set \{true, false\} of booleans. Then a filling propagator $J : \mathcal{E}_0 \to \mathcal{E}$ is built by adding to $\mathcal{E}_0$ a point $Ax$ and a potential monomorphism $m : Ax \to \text{Sen}$. This can be completed by a fractioning propagator $K : \mathcal{E} \to \overline{\mathcal{E}}$ and a filling propagator $\overline{J}$ such that $\overline{J} : K_0 \to K$ is a homomorphism of fractioning propagators, together with a $K$-domain $D$ such that $D_0 = U_D(D)$.

The point $K_0(\text{Sen})$ of $\overline{\mathcal{E}_0}$ stands for the set of sentences, the point $Ax$ of $\mathcal{E}$ for the set of axioms, and the point $K(Ax)$ of $\overline{\mathcal{E}}$ for the set of valid sentences.

Let $S$ be a $K$-specification and $S_0 = U_J(S)$ its support. Then $S(\text{Sen})$ is equal to $S_0(\text{Sen})$, and the image of $S(Ax)$ by $S(m)$ is a subset of $S_0(\text{Sen})$. Clearly, in this way, the category of $K$-specifications (up to isomorphisms) can be identified to the category of pairs $(S_0, V)$ where $S_0$ is a $K_0$-specification, $V$ is a subset of $S_0(\text{Sen})$, and the morphisms are straightforward.

This gives rise to an institution $I$ in the following way:
- $\text{Real}(\mathcal{E}_0)$ is the category of signatures of $I$,
- $\text{Mod} : \text{Real}(\mathcal{E}_0) \to \textbf{Set}$ is the contravariant functor of models of $I$,
- $\text{ev}_{\mathcal{E}_0} \circ F_{K_0} : \text{Real}(\mathcal{E}_0) \to \text{Set}$ is the functor of sentences of $I$,
- and for all signature $S_0$, all model $\tau$ of $S_0$ with values in $D_0$ and all sentence $s$ of $S_0$, the satisfaction relation between $\tau$ and $s$ holds if and only if $\tau$ satisfies (in the sense of diagrammatic specifications) the $K$-specification $S$ with support $S_0$ and with $s$ as its unique axiom.

Then the required satisfaction condition is easily checked.

In addition, such an institution, together with the notion of syntactic entailment (in the sense of diagrammatic specifications) gives rise to a logic in the sense of [Marti-Oliet and Meseguer, 1994].

In this context, we can clarify the relations between the diagrammatic notions of entailment $\rightarrow$ and consequence $\Rightarrow$, and the usual logical notions of entailment $\vdash$ and consequence $\models$.

Let $S_0$ be some fixed signature, and let $\varphi_1, \varphi_2, \ldots, \varphi_k$ and $\psi$ be sentences of $S_0$. Let $S$ be the specification with signature $S_0$ such that $S(Ax) = \{\varphi_1, \varphi_2, \ldots, \varphi_k\}$. Let $S'$ be the specification with signature $S_0$ such that $S'(Ax) = \{\varphi_1, \varphi_2, \ldots, \varphi_k, \psi\}$. Let $\sigma : S \rightarrow S'$ be the inclusion. Then clearly:

$$S \rightarrow S' \quad \text{if and only if} \quad \varphi_1, \varphi_2, \ldots, \varphi_k \vdash \psi,$$

$$S \Rightarrow_D S' \quad \text{if and only if} \quad \varphi_1, \varphi_2, \ldots, \varphi_k \models \psi.$$

References


