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A triangular central limit theorem under a new weak dependence condition

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Abstract

We use a new weak dependence condition from Doukhan and Louhichi (Stoch. Process. Appl. 1999, 84, 313–342) to provide a central limit theorem for triangular arrays; this result applies for linear arrays (as in Peligrad and Utev, Ann. Probab. 1997, 25(1), 443–456) and standard kernel density estimates under weak dependence. This extends on strong mixing and includes non-mixing Markov processes and associated or Gaussian sequences. We use Lindeberg method in Rio (Probab. Theory Related Fields 1996, 104, 255–282).

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1. Introduction

We use here a new weak dependence condition from Doukhan and Louhichi (1997,1999) in order to provide a central limit theorem (CLT in short) for triangular arrays; this result is used both for standard kernel density estimates and for general linear combinations of weakly dependent sequences, analogously to Peligrad and Utev (1997).

We work here under a fundamental *causality* assumption in order to be in position to use the Lindeberg method for dependent sequences developed by Rio (1995,1996).

Contrarily to previous authors, Rio does not use Bernstein blocks to prove a central limit theorem. The standard ways of proving such limit theorems for dependent random sequences are, after a decomposition into Bernstein blocks, to make use of the standard techniques for i.i.d. sequences; Lindeberg and Stein are the two techniques usually used (see references e.g. in Rio, 1996).

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For example, Bernstein blocks are used with Lindeberg method in Doukhan and Louhichi (1999): the results presented in this note clearly improve on CLTs stated in this basic paper under a more general non-causal frame.

A CLT is obtained for the kernel density estimates. The result is analogue to the one obtained for i.i.d. or mixing samples (see Rosenblatt, 1991; Robinson, 1983). Under our frame, it has to be noticed that we require the same assumptions here for such a CLT than to obtain minimax second order properties of the kernel estimates (see Doukhan and Louhichi, 1997).

Finally, the linear triangular CLT requires the same assumption as to prove the standard $\sqrt{n}$-CLT.

We now introduce our dependence frame; it is a variation on the definition in Doukhan and Louhichi (1999). Assume that, for convenient functions h and k,

$$\mathrm{Cov}(h(\text{'past'}), k(\text{'future'})),$$

converge to 0 as the distance between the 'past' and the 'future' converges to infinity. Here 'past' and 'future' refer to the values of some time series of interest. Asymptotically, this means that independence holds if we use a *determining* function class.

More precisely, E being some Euclidean space $\mathbb{R}^d$ endowed with its Euclidean norm $\|\,.\,\|$, we shall consider a sequence of E-valued random variables $(\xi_n)_{n\in\mathbb{N}}$. We define $\mathbb{L}^\infty$ as the set of measurable and bounded numerical functions on some space $\mathbb{R}^u$ and its norm is classically written $\|\bullet\|_\infty$. Moreover, let $u\in\mathbb{N}^*$ be a positive integer we endow the set $F=E^u$ with the norm $\|(x_1,\ldots,x_u)\|_F = \|x_1\| + \cdots + \|x_u\|$. Let now $h: F = E^u \to \mathbb{R}$ be a numerical function on F, we set

$$\mathrm{Lip}(h) = \sup_{x\neq y} \frac{|h(x)-h(y)|}{\|x-y\|_F},$$

the Lipschitz modulus of h. Define

$$\mathscr{L} = \bigcup_{u=1}^{\infty} \{h\in\mathbb{L}^\infty(\mathbb{R}^u,\mathbb{R});\ \|h\|_\infty \leqslant 1,\ \mathrm{Lip}(h)<\infty\}. \tag{1}$$

Definition 1. The sequence $(\xi_n)_{n\in\mathbb{N}}$ is s-weakly (resp. a-weakly) dependent, if for some sequence $\theta=(\theta_r)_{r\in\mathbb{N}}$ decreasing to zero at infinity and any $(u+2)$-tuple $(i_1,\ldots,i_u,j_1,j_2)$ with $i_1\leqslant\cdots\leqslant i_u < i_u+r\leqslant j_1\leqslant j_2$, and $h\in\mathbb{L}^\infty$ satisfies $\|h\|_\infty\leqslant 1$ and $k\in\mathscr{L}$,

$$|\mathrm{Cov}(h(\xi_{i_1},\ldots,\xi_{i_u}),k(\xi_{j_1},\xi_{j_2}))| \leqslant \mathrm{Lip}(k)\theta_r \tag{2}$$

and, respectively, for $h,k\in\mathscr{L}$

$$|\mathrm{Cov}(h(\xi_{i_1},\ldots,\xi_{i_u}),k(\xi_{j_1},\xi_{j_2}))| \leqslant \mathrm{Lip}(h)\mathrm{Lip}(k)\theta_r. \tag{3}$$

Weak dependence conditions are shown to hold in, either causal or noncausal frames in Doukhan and Louhichi (1997a). For this, consider also v-tuples $(j_1,\ldots,j_v)$ with $i_1\leqslant\cdots\leqslant i_u<i_u+r\leqslant j_1\leqslant\cdots\leqslant j_v$, then such weak dependence conditions are defined for functions h and k defined on E^u and E^v, respectively, through inequalities

$$|\mathrm{Cov}(h(\xi_{i_1},\ldots,\xi_{i_u}),k(\xi_{j_1},\ldots,\xi_{j_v}))| \leqslant v\,\mathrm{Lip}(k)\theta_r \tag{4}$$

if $k\in\mathscr{L}$ and $\|h\|_\infty\leqslant 1$ or

$$|\mathrm{Cov}(h(\xi_{i_1},\ldots,\xi_{i_u}),k(\xi_{j_1},\ldots,\xi_{j_v}))| \leqslant \min(u,v)\mathrm{Lip}(h)\mathrm{Lip}(k)\theta_r \tag{5}$$

if $h,k\in\mathscr{L}$. Strong mixing defined by Rosenblatt (see e.g., Doukhan, 1994) is a variation of such definitions (see Doukhan and Louhichi, 1999); however mixing refers to σ-algebras rather than to random variables. For completeness, we now recall some examples adapted from Doukhan and Louhichi (1999), where noncausal cases are also considered:

Definition 2. Let $(\eta_n)_{n \in \mathbb{Z}}$ be a stationary sequence of real valued r.v's and F be a measurable function defined on $\mathbb{R}^{\mathbb{N}}$. The stationary sequence $(\xi_n)_{n \in \mathbb{Z}}$ defined by $\xi_n = F(\eta_n, \eta_{n-1}, \eta_{n-2}, \ldots)$ is called a causal Bernoulli shift. We denote $(\delta_k)_{k \in \mathbb{N}}$ any nonnegative sequence such that

$$\mathbb{E}|F(\eta_0, \eta_{-1}, \eta_{-2}, \ldots) - F(\eta_0, \ldots, \eta_{-r}, 0, 0, \ldots)| \leqslant \delta_r.$$

Causal shifts with i.i.d. innovations $(\eta_k)_{k \in \mathbb{Z}}$ satisfy (4) with $\theta_r = 2\delta_r$ (see Rio, 1996). Examples of such situations follow:

- The example of the nonmixing stationary Markov chain with i.i.d. Binomial innovations $(\eta_t)_{t \in \mathbb{Z}}$, $\xi_n = (\xi_{n-1} + \eta_n)/2$ satisfies $\delta_r = \mathcal{O}(2^{-r})$; its marginal distribution is uniform on $[0, 1]$.
- The real-valued functional autoregressive model $\xi_t = r(\xi_{t-1}) + \eta_t$, $r : \mathbb{R} \to \mathbb{R}$. If $|r(u) - r(u')| \leqslant c|u - u'|$ for some $0 \leqslant c < 1$ and for all $u, u' \in \mathbb{R}$, and if the i.i.d. innovation process $(\eta_t)_{t \in \mathbb{Z}}$ satisfies $\mathbb{E}\|\xi_0\| < \infty$, then s-dependence holds with $\theta_r = \delta_r = Cc^r$ for some constant $C > 0$.
- Chaotic expansions associated with the discrete chaos generated by the sequence $(\eta_t)_{t \in \mathbb{Z}}$. In a condensed formulation we write, $F(x) = \sum_{k=0}^{\infty} F_k(x)$, $x \in \mathbb{R}^{\mathbb{N}}$ for

  $$F_k(x) = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_k=0}^{\infty} a^{(k)}_{j_1,\ldots,j_k} x_{j_1} x_{j_2} \ldots x_{j_k}, \quad k \geqslant 1,$$

  where $F_k(x)$ denotes the kth-order chaos contribution and $F_0(x) = a_0^{(0)}$ is only a centering constant. In short we write in the vectorial notation, $F_k(x) = \sum_{j \in \mathbb{N}^k} a_j^{(k)} x_j$. Processes associated with a finite number of chaos (i.e. $F_k \equiv 0$ if $k > k_0$ for some $k_0 \in \mathbb{N}$) are also called *Volterra processes*. A simple and general condition for $\mathbb{L}^1$-convergence of this expansion, still written in a condensed notation, is $\sum_{k=0}^{\infty} \{\sum_{j \in \mathbb{N}^k} |a_j^{(k)}| \mathbb{E}|\xi_0|^k\} < \infty$. This condition allows to define the distribution of such shift processes. A suitable bound for δ_r is then

  $$\delta_r = \sum_{k=0}^{\infty} \left\{ \sum_{j \in \mathbb{N}^k;\, \|j\|_\infty > r} \left| a_j^{(k)} \right| \mathbb{E}|\xi_0|^k \right\} < \infty.$$

- For example, linear processes $\xi_n = \sum_{k=0}^{\infty} a_k \eta_{n-k}$ which include ARMA models are those with $F_k(x) \equiv 0$ for all $k > 1$. A first choice is $\delta_r = \mathbb{E}|\eta_0| \sum_{k > r} |a_k|$ for the linear process with i.i.d. innovations such that $\mathbb{E}|\eta_0| < \infty$.
  For centered and $\mathbb{L}^2$ innovations, another choice is $\delta_r = \sqrt{\mathbb{E}|\eta_0|^2 \sum_{k > r} |a_k|^2}$.
- The simple bilinear process with the recurrence equation $\xi_t = a\xi_{t-1} + b\xi_{t-1}\eta_{t-1} + \eta_t$. Such processes are associated with the chaotic representation in

  $$F(x) = \sum_{j=1}^{\infty} x_j \prod_{s=0}^{j-1} (a + bx_s), \quad x \in \mathbb{R}^{\mathbb{Z}}.$$

  If $c = \mathbb{E}|a + b\xi_0| < 1$ then $\delta_r = \theta_r = c^r(r + 1)/(c - 1)$ has a geometric decay rate.

Definition 3 (Esary et al., 1967)**.** The sequence $(\xi_n)_{n \in \mathbb{Z}}$ is associated if for all coordinatewise increasing real-valued functions h and k on $\mathbb{R}^A$ $(A \subset \mathbb{Z})$, $\mathrm{Cov}(h(\xi_A), k(\xi_A)) \geqslant 0$, if $E[h^2(\xi_A) + k^2(\xi_A)] < \infty$.

Associated sequences satisfy (5) with $\theta_r = \sup_{t \in \mathbb{Z}} \sum_{k \geqslant r} |\mathrm{Cov}(\xi_t, \xi_{t+k})|$. Absolute values are useless here, but Gaussian sequences also satisfy this condition with them. Hence combinations of such independent processes yield examples of weak dependent sequences which are neither Gaussian nor associated.

The paper is organized as follows. Our main results are stated with their applications in Section 2, and Section 3 is devoted to prove the main results.

2. Main results

This paper is concerned with triangular arrays $(X_{n,k})_{k=1,\dots,k_n}$ for $n=1,2,\dots$ defined through an E-valued weakly dependent sequence $(\xi_n)_{n\in\mathbb{N}}$ by numerical Lipschitz functions $g_{n,k}$ defined on E for $k=1,\dots,k_n$ and $n=1,2,\dots,$ we assume that the sequence of integers k_n increases to infinity with n. We set

$$X_{n,k}=g_{n,k}(\xi_k) \quad \text{and} \quad S_n=X_{n,1}+\cdots+X_{n,k_n}.$$

We assume in the following that $\mathbb{E}X_{n,k}=0$. Now let $S_{k,n}=X_{n,1}+\cdots+X_{n,k}$ for $1\leqslant k\leqslant k_n$, we also suppose that the exist constants $\sigma,\alpha>0$ such that

$$\lim_{n\to\infty}\operatorname{Var}S_n=\sigma^2>0 \quad \text{and} \quad v_{k,n}=\operatorname{Var}S_{k,n}-\operatorname{Var}S_{k-1,n}\geqslant\frac{\alpha}{n}, \tag{6}$$

for each $k\in\{1,\dots,k_n\}$ and for any integer n.

We shall also set

$$\lambda_n=\sup_{1\leqslant k\leqslant k_n}\operatorname{Lip}(g_{k,n}), \quad M_n=\sup_{1\leqslant k\leqslant k_n}\|g_{k,n}\|_\infty,$$
$$\delta_n=\sup_{1\leqslant k\leqslant k_n}\mathbb{E}|X_{k,n}| \quad \text{and} \quad \Delta_n=\sup_{1\leqslant k\neq l\leqslant k_n}\mathbb{E}|X_{k,n}X_{l,n}|. \tag{7}$$

We are now in a position to state the two main results of this paper.

Theorem 1. *Assume that the E-valued sequence satisfies the s-weak dependence condition and the triangular array* $(X_{n,k})_{1\leqslant k\leqslant k_n}$ *defined as before satisfies assumption* (6), *then if*

$$(k_nM_n+k_n^{2/3})M_n\delta_n\to0, \quad k_nM_n\sum_{p=1}^{k_n}\min(\lambda_n\theta_p,\Delta_n)\to0, \quad k_n\sum_{p=1}^{k_n}\min(M_n\lambda_n\theta_p,\Delta_n)\to0,$$

as $n\to\infty$, *we obtain*

$$S_n\to_{n\to\infty}\mathcal{N}(0,\sigma^2) \quad \textit{in distribution.}$$

Theorem 2. *Assume that the E-valued sequence satisfies the a-weak dependence condition and the triangular array* $(X_{n,k})_{1\leqslant k\leqslant k_n}$ *defined as before satisfies assumption* (6), *then if*

$$(k_nM_n+k_n^{2/3})M_n\delta_n\to0, \quad k_nM_n\sum_{p=1}^{k_n}\min(\lambda_n^2\theta_p,\Delta_n)\to0, \quad k_n\sum_{p=1}^{k_n}\min(\lambda_n^2\theta_p,\Delta_n)\to0,$$

as $n\to\infty$, *we obtain*

$$S_n\to_{n\to\infty}\mathcal{N}(0,\sigma^2) \quad \textit{in distribution.}$$

Applications of those results given in Section 3 will prove that they lead to real improvement of previous results. They also lead to completely new results extending e.g. in Peligrad and Utev (1997).

2.1. Linear triangular arrays

The random variables ξ_k are supposed to be uniformly bounded, real valued and centered at expectation and here

$$g_{n,k}(x)=a_{n,k}x,$$

hence setting $b_n=\sup_{1\leqslant k\leqslant k_n}|a_{n,k}|$ we obtain, $\lambda_n\leqslant Cb_n$, $M_n\leqslant Cb_n$, $\delta_n\leqslant Cb_n$, $\Delta_n\leqslant Cb_n^2$, for some constant $C>0$. We deduce the following result:

Corollary 1. *Assume that the* $\mathbb{R}$*-valued sequence satisfies either the s-weak dependence condition or the a-weak dependence condition and assumptions* (6). *If moreover*, $k_n b_n^2 \to 0$, *as* $n \to \infty$ *and* $\sum_{p=1}^{\infty} \theta_p < \infty$ *then*

$$S_n \to \mathcal{N}(0,\sigma^2) \quad \textit{in distribution, as } n \to \infty.$$

Notice first that as in Peligrad and Utev (1997), we need $b_n \to 0$. Moreover, those authors also assume the first condition in (6); the second one is however not always satisfied (as for the example of regression with a fixed design described without proof in the latter paper).

The proof also yields the (standard) CLT $n^{-1/2}\sum_1^n \xi_k \to \mathcal{N}(0,\sigma^2)$ for a stationary and bounded weakly dependent sequence with $\sigma^2 > 0$ and $\sum_{p=0}^{\infty} \theta_p < \infty$.

2.2. Density estimation

Let u be some numerical function with integral 1, Lipschitzian and rapidly convergent to 0 at infinity, for simplicity, we assume here that it is compactly supported; one classically (see e.g. Rosenblatt, 1991) defines kernel density estimates for the marginal density of the process (ξ_n) by setting for some fixed sequence (h_n) of positive real numbers such that $h \to 0$ (for clarity we write h instead h_n) and $nh \to 0$ as $n \to \infty$,

$$\hat{f}(x) = f_{n,h_n}(x) \quad \text{with} \quad f_{n,h}(x) = \frac{1}{nh}\sum_{k=1}^{n} u\left(\frac{x-\xi_k}{h}\right).$$

We assume from now on, that the marginal density of X_n exists and we denote it f. The study of the bias of this estimate is purely analytical and does not depend on the dependence properties of the sequence (ξ_n) as it is noticed in Rosenblatt (1991). We thus restrict our attention to the centered estimation process, written as

$$Z_n(x) := Z_{n,h_n}(x) = \sqrt{nh_n}(\hat{f}(x) - \mathbb{E}\hat{f}(x)).$$

Let $x_1,\ldots,x_l \in \mathbb{R}$, if one wants to know the asymptotic behaviour in distribution of the vector $(Z_n(x_1),\ldots, Z_n(x_l))$, it is sufficient to use the previous theorems with $k_n = n$ and, $X_{n,k} = (1/\sqrt{nh})\sum_{j=1}^{l} s_j u(\xi_k - x_j/h)$, for arbitrary numbers $s_1,\ldots,s_l \in \mathbb{R}$, to see that this example enters the frame of a general triangular array.

Corollary 2. *Assume that the previous s-weak dependence* (*resp. a-*) *condition holds for the stationary sequence* $(\xi_n)_{n\in\mathbb{N}}$ *with for some positive* $a < \frac{1}{3}$ (*resp.* $a < \frac{1}{4}$) $\sum_{p=1}^{\infty} \theta_p^a < \infty$, *then the finite-dimensional marginals* $(Y_n(x_1),\ldots,Y_n(x_l))$, *of the process* $Y_n(x) \equiv Z_n(x)/\sqrt{f(x)\int_{-\infty}^{\infty} u^2(t)\,\mathrm{d}t}$ *converge in distribution to an* $\mathcal{N}(0,I_l)$ *random variable if we assume moreover that* $f(x_1) \neq 0,\ldots,f(x_l) \neq 0$, *that* ξ_0*'s marginal admits a continuous marginal density* f *and the marginal densities* $f_k(x,y)$ *of the bivariate random variables* (ξ_0,ξ_k) *exist for any* $k > 0$ *and satisfy* $\sup_{k>0}\sup_{(x,y)\in\mathbb{R}^2} f_k(x,y) < \infty$.

Remarks

- These conditions hold, respectively, if $\theta_r = \mathcal{O}(r^{-a})$ for some $a > 3$ (resp. $a > 4$).
- This result improves on a previous result in Doukhan and Louhichi (1997) (see also Isha and Prakasa, 1995 or Chanda and Ruymgart, 1990), e.g. under association we need $\mathrm{Cov}(\xi_0,\xi_r) = \mathcal{O}(r^{-a})$ for $a > 5$ while the previous result was obtained assuming $a > 12$ and for causal shifts it was needed that $\theta_r = \mathcal{O}(r^{-a})$ for some $a > \max\{9, \frac{3}{2}(1+\delta^{-1})\}$ if $h \sim n^{-\delta}$. In both cases, the result seems to be new.
- For strongly mixing sequences, the condition $\alpha_n = \mathcal{O}(n^{-a})$ for $a > 1$ ensures this CLT as proved by Robinson (1983) (and also Ango Nze and Doukhan, 1996); this assumptions is of a different nature, e.g. linear processes satisfy the mixing conditions (under additional regularity conditions; see Doukhan, 1994, Chapter 2.3) the decay rate of the coefficients are there more restrictive.

Up to constants, we obtain $\lambda_n = 1/(h\sqrt{nh})$, $M_n = \lambda_n h$, $\Delta_n = h/n$, and $\delta_n = h/\sqrt{nh}$. Conditions (6) follow by standard arguments from assumptions on marginal densities and from $\sum_{p=0}^{\infty}\theta_p^a < \infty$ for $3a < 1$ (resp. for $4a < 1$) (see e.g. Doukhan and Louhichi, 1997 or Ango Nze and Doukhan, 1996). We now split the proof in the cases considered.

2.2.1. s-dependence

Now $nM_n^2\delta_n + n^{2/3}M_n\delta_n = 1/(\sqrt{nh}) + (4h^{1/3}/(nh)^{1/3})$, so that we just need $nh \to_{n\to\infty} \infty$. Let $a \in]0,1]$,

$$nM_n\sum_{p=1}^{\infty}\min(\lambda_n\theta_p,\Delta_n) \leqslant \sum_{p=1}^{\infty}\left(\frac{\theta_p}{h^2}\right)^a\left(\sqrt{\frac{h}{n}}\right)^{1-a} = \frac{1}{(nh)^{(1-a)/2}}h^{1-3a}\sum_{p=1}^{\infty}\theta_p^a.$$

We need that for some $a \leqslant \frac{1}{3}$, $\sum_{p=1}^{\infty}\theta_p^a < \infty$.

$$n\sum\min(\Delta_n, M_n\lambda_n\theta_p) \leqslant \sum_{p=1}^{\infty}\min\left(h, \frac{n\theta_p}{nh^2}\right) \leqslant \sum_{p=1}^{\infty}\theta_p^a h^{-2a}h^{1-3a}.$$

If for some $0 < a < \frac{1}{3}$, $\sum_{p=1}^{\infty}\theta_p^a < \infty$, the previous expression tends to 0 when $n \to \infty$.

2.2.2. a-dependence

Now, $nM_n^2\delta_n = 1/\sqrt{nh} \to 0$ as $n \to \infty$. We also consider

$$nM_n\sum_{p=1}^{\infty}\min(\lambda_n^2\theta_p,\Delta_n) = \sum_{p=1}^{\infty}\min\left(\sqrt{\frac{h}{n}}, \frac{\theta_p}{n^{1/2}h^{7/2}}\right) \leqslant \frac{1}{\sqrt{nh}}h^{1-4a}\sum_{p=1}^{\infty}\theta_p^a \quad \text{for } a \in]0,1].$$

So we need that there exists $a \leqslant \frac{1}{4}$ with $\sum_{p=1}^{\infty}\theta_p^a < \infty$.

$$n\sum_{p=1}^{\infty}\min(\lambda_n^2\theta_p,\Delta_n) = \sqrt{nh}\sum_{p=1}^{\infty}\min\left(\sqrt{\frac{h}{n}}, \frac{\theta_p}{n^{1/2}h^{7/2}}\right) \leqslant h^{1-4a}\sum_{p=1}^{\infty}\theta_p^a.$$

If there is some positive $a < \frac{1}{4}$ with $\sum_{p=1}^{\infty}\theta_p^a < \infty$, we conclude the proof. □

3. Proofs

3.1. Proof of Theorem 1

The proof is a variation of Lindeberg method after Rio (1995). Consider a bounded thrice differentiable function $h : \mathbb{R} \to \mathbb{R}$ with continuous and bounded derivatives. Set $C_j = \|h^{(j)}\|_\infty$, for $j = 0, 1, 2, 3$. Also consider $\sigma_n^2 = \operatorname{Var} S_n$. Set, for some standard Gaussian r.v. η, $\Delta_n(h) = \mathbb{E}(h(S_n) - h(\sigma_n\eta))$. The theorem will follow from assumptions (6), if we prove that

$$\lim_{n\to\infty}\Delta_n(h) = 0.$$

Recall that $v_{k,n} > 0$ for each k and set $Y_{n,k} \sim \mathcal{N}(0, v_{n,k})$. The sequence $(Y_{n,k})_{1\leqslant k\leqslant k_n, n\geqslant 1}$ is assumed to be independent and independent of the sequence $(\xi_k)_{k\in\mathbb{N}}$, and set, if $1 \leqslant k \leqslant k_n$, $T_{k,n} = \sum_{j=k+1}^{k_n} Y_{n,j}$, empty sums are, as usual, set equal to 0. We are in position to use Rio's decomposition

$$\Delta_n(h) = \sum_{k=2}^{k_n}\Delta_{k,n}(h) \tag{8}$$

with $\Delta_{k,n}(h) = \mathbb{E}(h(S_{k-1,n} + X_{n,k} + T_{k,n}) - h(S_{k-1,n} + Y_{n,k} + T_{k,n}))$.

The function $x \to h_{k,n}(x) = \mathbb{E}\,h(x+T_{k,n})$ has the same derivability properties as h, e.g. for $0 \leqslant j \leqslant 3$, $\|h_{k,n}^{(j)}\| \leqslant C_j$; now, from independence of the Gaussian r.v. $T_{k,n}$ and the process $(\xi_n)_{n\in\mathbb{N}}$, we write $\Delta_{k,n}(h) = \Delta_{k,n}^{(1)}(h) - \Delta_{k,n}^{(2)}(h)$, with

$$\Delta_{k,n}^{(1)}(h) = \mathbb{E}\,h_{k,n}(S_{k-1,n} + X_{n,k}) - \mathbb{E}\,h_{k,n}(S_{k-1,n}) - \frac{v_{k,n}}{2}\mathbb{E}\,h''_{k,n}(S_{k-1,n}),$$

$$\Delta_{k,n}^{(2)}(h) = \mathbb{E}\,h_{k,n}(S_{k-1,n} + Y_{n,k}) - \mathbb{E}\,h_{k,n}(S_{k-1,n}) - \frac{v_{k,n}}{2}\mathbb{E}\,h''_{k,n}(S_{k-1,n}).$$

Bound of $\Delta_{k,n}^{(2)}(h)$. Using Taylor expansion yields for some random variable valued $\rho_{n,k} \in (0,1)$: $\Delta_{k,n}^{(2)}(h) = \mathbb{E}\,h'_{k,n}(S_{k-1,n})Y_{n,k} + \frac{1}{2}\mathbb{E}\,h''_{k,n}(S_{k-1,n})(Y_{n,k}^2 - v_{n,k}) + \frac{1}{6}\mathbb{E}\,h_{k,n}^{(3)}(S_{k-1,n} + \rho_{n,k}Y_{n,k})Y_{n,k}^3$. From the independence of the Gaussian sequence, $|\Delta_{k,n}^{(2)}(h)| \leqslant (C_3/6)\mathbb{E}|Y_{n,k}|^3$, hence $|\Delta_{k,n}^{(2)}(h)| \leqslant (2C_3 v_{k,n}^{3/2}/3\sqrt{2\pi})$. Now $v_{k,n} = \operatorname{Var} X_{n,k} + 2\sum_{j=1}^{k-1}\operatorname{Cov}(X_{n,j}, X_{n,k})$, hence $v_{k,n} \leqslant M_n\delta_n + 2\sum_{j=1}^{k_n-1}\min(\lambda_n M_n\theta_j, \Delta_n)$. We thus need

$$k_n^{2/3}\left[M_n\delta_n + 2\sum_{j=1}^{k-1}\min(\lambda_n M_n\theta_j, \Delta_n)\right] \to_{n\to\infty} 0. \tag{9}$$

Bound of $\Delta_{k,n}^{(1)}(h)$. Set $\Delta_{k,n}^{(1)}(h) = \mathbb{E}\delta_{k,n}^{(1)}(h)$ we write, again with some random $\tau_{k,n} \in (0,1)$, $\delta_{k,n}^{(1)}(h) = h'_{k,n}(S_{k-1,n})X_{n,k} + \frac{1}{2}h''_{k,n}(S_{k-1,n})(X_{n,k}^2 - v_{n,k}) + \frac{1}{6}(h_{k,n}^{(3)}(S_{k-1,n} + \tau_{n,k}X_{n,k})X_{n,k}^3)$. We analyze separately the term in the previous expression

$$\frac{1}{6}|\mathbb{E}\,h_{k,n}^{(3)}(S_{k-1,n} + \tau_{n,k}X_{n,k})X_{n,k}^3| \leqslant \frac{C_3}{6}M_n^2\delta_n. \tag{10}$$

To estimate the median term, we write $\operatorname{Cov}(h''_{k,n}(S_{k-1,n}), X_{n,k}^2) = \sum_{j=1}^{k-1}\operatorname{Cov}((h''_{k,n}(S_{j,n}) - h''_{k,n}(S_{j-1,n}), X_{n,k}^2)$, hence

$$|\operatorname{Cov}(h''_{k,n}(S_{k-1,n}), X_{n,k}^2)| \leqslant \max(2C_2,\, C_3)M_n\sum_{j=1}^{k-1}\min(\lambda_n\theta_j, \Delta_n). \tag{11}$$

$$|\operatorname{Cov}(h'_{k,n}(S_{i,n}) - h'_{k,n}(S_{i-1,n}), X_{n,k})| \leqslant C\min(M_n\lambda_n\theta_{k-i}, \Delta_n) \tag{12}$$

and

$$|Eh''_{k,n}(S_{k-1,n})EX_{n,i}X_{n,k}| \leqslant C\min(M_n\lambda_n\theta_{k-i}, \Delta_n). \tag{13}$$

Adding (12) and (13) and summing up the expression for all i we get

$$\left|\mathbb{E}\,h'_{k,n}(S_{k-1,n})X_{n,k} - \mathbb{E}\,h''_{k,n}(S_{k-1,n})\sum_{i=1}^{k-1}\mathbb{E}X_{n,i}X_{n,k}\right| \leqslant C\sum_{p=1}^{k-1}\min(M_n\lambda_n\theta_p, \Delta_n). \tag{14}$$

We add Eqs. (10), (11) and (14) to obtain

$$|\Delta_{k,n}^{(1)}(h)| \leqslant \max(2C_2,\, C_3)\left[M_n^2\delta_n + M_n\sum_{p=1}^{k-1}\min(\lambda_n\theta_p, \Delta_n) + \sum\min(M_n\lambda_n\theta_p, \Delta_n)\right]. \tag{15}$$

Now we sum for all k to conclude

$$\left|\sum_{k=2}^{k_n}\Delta_{k,n}^{(1)}(h)\right| \leqslant C\left[k_nM_n^2\delta_n + k_nM_n\sum_{p=1}^{\infty}\min(\lambda_n\theta_p, \Delta_n) + k_n\sum_{p=1}^{\infty}\min(M_n\lambda_n\theta_p, \Delta_n)\right]. \tag{16}$$

3.2. Proof of Theorem 2

Only replace, respectively, Eqs. (11)–(14) by

$$|\mathrm{Cov}(h''_{k,n}(S_{k-1,n}), X_{n,k}^2)| \leqslant CM_n \sum_{j=1}^{k-1} \min(\lambda_n^2\theta_j, \Delta_n),$$

$$|\mathrm{Cov}(h'_{k,n}(S_{i,n}) - h'_{k,n}(S_{i-1,n}), X_{n,k})| \leqslant C \min(\Delta_n, \lambda_n^2\theta_{k-i}),$$

$$|\mathbb{E}\, h''_{k,n}(S_{k-1,n})\mathbb{E} X_{n,i}X_{n,k}| \leqslant C \min(\Delta_n, \lambda_n^2\theta_{k-i})$$

and

$$\left|\mathbb{E}\, h'_{k,n}(S_{k-1,n})X_{n,k} - \mathbb{E}\, h''_{k,n}(S_{k-1,n}) \sum_{i=1}^{k-1} \mathbb{E} X_{n,i}X_{n,k}\right| \leqslant C \sum_{p=1}^{k-1} \min(\lambda_n^2\theta_{k-i}, \Delta_n).$$

Then we replace (15) by

$$|\Delta_{k,n}^{(1)}(h)| \leqslant C\left[M_n^2\delta_n + M_n \sum_{p=1}^{k-1} \min(\lambda_n^2\theta_p, \Delta_n) + \sum_{p=1}^{k-1} \min(\lambda_n^2\theta_p, \Delta_n)\right].$$

Finally we replace (16) by

$$\left|\sum_{k=2}^{k_n} \Delta_{k,n}^{(1)}(h)\right| \leqslant Ck_nM_n^2\delta_n + k_nM_n \sum_{p=1}^{\infty} \min(\lambda_n^2\theta_p, \Delta_n) + \sum_{p=1}^{\infty} \min(\lambda_n^2\theta_p, \Delta_n).$$

References

Ango Nze, P., Doukhan, P., 1996. Non-parametric Minimax estimation in a weakly dependent framework Is: Quadratic properties. Math. Meth. Statist. 5 (4), 404–423.

Chanda, K.C., Ruymgart, F.H., 1990. General linear processes: a property of the empirical process applied to density and mode estimation. J. Time Ser. Anal. 11 (3), 185–199.

Doukhan, P., 1994. Mixing: Properties and Examples, Lecture Notes in Statistics, Vol. 85. Springer, Berlin.

Doukhan, P., Louhichi, S., 1997. Functional estimation of a density under a new weak dependence condition. Université de Cergy Pontoise, Prépublication 10.97 (a short version of this appeared in C.R.A.S. Paris (1998) 327, 989–992).

Doukhan, P., Louhichi, S., 1999. A new weak dependence condition and applications to moment inequalities. Stoch. Process. Appl. 84, 313–342.

Esary, J., Proschan, F., Walkup, D., 1967. Association of random variables with applications. Ann. Math. Statist. 38, 1466–1476.

Isha, B., Prakasa, R., 1995. Kernel-type density and failure rate estimation for associated sequences. Ann. Inst. Statist. Math. 47, 253–266.

Peligrad, M., Utev, S., 1997. Central limit theorem for linear processes. Ann. Probab. 25 (1), 443–456.

Rio, E., 1995. About the Lindeberg method for strongly mixing sequences. ESAIM, Probab. Statist. 1, 35–61.

Rio, E., 1996. Sur le théorème de Berry–Esseen pour les suites faiblement dépendantes. Probab. Theory Related Fields 104, 255–282.

Robinson, P.M., 1983. Non parametric estimators for time series. J. Time Ser. Anal. 4 (3), 185–207.

Rosenblatt, M., 1991. Stochastic curve estimation. NSF-CBMS Regional Conference Series in Probability and Statistics, Vol. 3.

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Erratum de **"A triangular central limit theorem under a new weak dependence condition"**, 17 septembre 2001

1 Résultats principaux

Theorem 1 *Assume that the E-valued sequence satisfies the 's'-weak dependence condition and the triangular array $(X_{n,k})_{1\le k\le k_n}$ defined as before satisfies assumption (6), then if*

$$(k_nM_n + k_n^{\frac{2}{3}})M_n\delta_n \to 0, \quad k_nM_n\sum_{p=1}^{\infty}\min(\lambda_n\theta_p, \Delta_n+\delta_n^2)\to 0, \quad k_n\sum_{p=1}^{\infty}\min(M_n\lambda_n\theta_p, \Delta_n)\to 0,$$

as $n\to\infty$ we obtain

$$S_n \to_{n\to\infty} \mathcal{N}(0,\sigma^2), \quad \textit{in distribution.}$$

Theorem 2 *Assume that the E-valued sequence satisfies the 'a'-weak dependence condition and the triangular array $(X_{n,k})_{1\le k\le k_n}$ defined as before satisfies assumption (6), then if*

$$(k_nM_n + k_n^{\frac{2}{3}})M_n\delta_n \to 0, \quad k_nM_n\sum_{p=1}^{\infty}\min(\lambda_n^2\theta_p, \Delta_n+\delta_n^2)\to 0, \quad \sum_{p=1}^{\infty}\min(\lambda_n^2\theta_p, \Delta_n)\to 0,$$

as $n\to\infty$ we obtain

$$S_n \to_{n\to\infty} \mathcal{N}(0,\sigma^2), \quad \textit{in distribution.}$$

D'autre part dans l'hypothèse (6), on peut remplacer $v_{k,n} = \operatorname{Var} S_{k,n} - \operatorname{Var} S_{k-1,n} \ge \alpha > 0$ par $\exists\, n_0 \in \mathbb{N}^*/v_{k,n} > 0\ \forall n \ge n_0,\ \forall 1\le k\le n$.

Il faut aussi remplacer la remarque "This result also yields the (standard) CLT $n^{-1/2}\sum_1^n \xi_k \longrightarrow \mathcal{N}(0,\sigma^2)$ for a stationary and bounded weakly dependent sequence with $\sigma^2>0$ and $\sum_{p=0}^{\infty}\theta_p<\infty$" de la page 65 par "A variation on the proof of the main result yields the standard CLT".

On remplace aussi 2.2.1. et 2.2.2. par

s-dependence:

Now $nM_n^2\delta_n + n^{\frac{2}{3}}M_n\delta_n = \frac{1}{\sqrt{nh}} + \frac{1}{n^{\frac{1}{3}}}$, so that we just need $nh \to_{n\to\infty}\infty$. Let $a\in]0,1]$,

$$nM_n\sum_{p=1}^{\infty}\min(\lambda_n\theta_p, \Delta_n+\delta_n^2) \le \sum_{p=1}^{\infty}\left(\frac{\theta_p}{h^2}\right)^a\left(\sqrt{\frac{h}{n}}\right)^{1-a} = \frac{1}{(nh)^{\frac{1-a}{2}}}h^{1-3a}\sum_{p=1}^{\infty}\theta_p^a.$$

We need that for some $a\le\frac{1}{3}$, $\sum_{p=1}^{\infty}\theta_p^a<\infty$.

$$n\sum\min(\Delta_n, M_n\lambda_n\theta_p) \le \sum_{p=1}^{\infty}\min\left(h, \frac{n\theta_p}{nh^2}\right) \le \sum_{p=1}^{\infty}\theta_p^a h^{-2a}h^{1-a}.$$

If for some $0<a<\frac{1}{3}$, $\sum_{p=1}^{\infty}\theta_p^a<\infty$, the previous expression tends to 0 when $n\to\infty$.

a-dependence: Now, $nM_n^2\delta_n = \frac{1}{\sqrt{nh}}\to 0$ as $n\to\infty$. We also consider

$$nM_n\sum_{p=1}^{\infty}\min(\lambda_n^2\theta_p, \Delta_n+\delta_n^2) = \sum_{p=1}^{\infty}\min\left(\sqrt{\frac{h}{n}}, \frac{\theta_p}{n^{\frac{1}{2}}h^{\frac{7}{2}}}\right) \le \frac{1}{\sqrt{nh}}h^{1-4a}\sum_{p=1}^{\infty}\theta_p^a \quad \text{for } a\in]0,1].$$

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So we need that exists $a \leq \frac{1}{4}$ with $\sum_{p=1}^{\infty} \theta_p^a < \infty$.

$$n \sum_{p=1}^{\infty} \min(\lambda_n^2 \theta_p, \Delta_n) = \sqrt{nh} \sum_{p=1}^{\infty} \min\left(\sqrt{\frac{h}{n}}, \frac{\theta_p}{n^{\frac{1}{2}} h^{\frac{7}{2}}}\right) \leq h^{1-4a} \sum_{p=1}^{\infty} \theta_p^a.$$

If there is some positive $a < \frac{1}{4}$ with $\sum_{p=1}^{\infty} \theta_p^a < \infty$, we conclude the proof.

2 Preuves des résultats principaux

On remplace, dans la preuve du Théorème 1, (11) page 67 par:

$$|\mathrm{Cov}(h''_{k,n}(S_{k-1,n}), X_{n,k}^2)| \leq \max(2C_2\ ,\ C_3) M_n \sum_{j=1}^{k-1} \min(\lambda_n \theta_j, \Delta_n + \delta_n^2). \tag{1}$$

On remplace (15) et (16) page 67 par:

$$|\Delta_{k,n}^{(1)}(h)| \leq \max(2C_2\ ,\ C_3) \left[M_n^2 \delta_n + M_n \sum_{p=1}^{k-1} \min(\lambda_n \theta_p, \Delta_n + \delta_n^2) + \sum \min(M_n \lambda_n \theta_p, \Delta_n) \right] \tag{2}$$

et

$$\left| \sum_{k=2}^{k_n} \Delta_{k,n}^{(1)}(h) \right| \leq C \left[k_n M_n^2 \delta_n + k_n M_n \sum_{p=1}^{\infty} \min(\lambda_n \theta_p, \Delta_n + \delta_n^2) + k_n \sum_{p=1}^{\infty} \min(M_n \lambda_n \theta_p, \Delta_n) \right]. \tag{3}$$

Dans la preuve du Théorème 2, on remplace

$$|\mathrm{Cov}(h''_{k,n}(S_{k-1,n}), X_{n,k}^2)| \leq C M_n \sum_{j=1}^{k-1} \min(\lambda_n^2 \theta_j, \Delta_n),$$

par

$$|\mathrm{Cov}(h''_{k,n}(S_{k-1,n}), X_{n,k}^2)| \leq C M_n \sum_{j=1}^{k-1} \min(\lambda_n^2 \theta_j, \Delta_n + \delta_n^2).$$

On remplace

$$|\Delta_{k,n}^{(1)}(h)| \leq C[M_n^2 \delta_n + M_n \sum_{p=1}^{k-1} \min(\lambda_n^2 \theta_p, \Delta_n) + \sum_{p=1}^{k-1} \min(\lambda_n^2 \theta_p, \Delta_n)]$$

par

$$|\Delta_{k,n}^{(1)}(h)| \leq C[M_n^2 \delta_n + M_n \sum_{p=1}^{k-1} \min(\lambda_n^2 \theta_p, \Delta_n + \delta_n^2) + \sum_{p=1}^{k-1} \min(\lambda_n^2 \theta_p, \Delta_n)].$$

Enfin on remplace

$$|\sum_{k=2}^{k_n} \Delta_{k,n}^{(1)}(h)| \leq C k_n M_n^2 \delta_n + k_n M_n \sum_{p=1}^{\infty} \min(\lambda_n^2 \theta_p, \Delta_n) + \sum_{p=1}^{\infty} \min(\lambda_n^2 \theta_p, \Delta_n)$$

par

$$|\sum_{k=2}^{k_n} \Delta_{k,n}^{(1)}(h)| \leq C k_n M_n^2 \delta_n + k_n M_n \sum_{p=1}^{\infty} \min(\lambda_n^2 \theta_p, \Delta_n + \delta_n^2) + \sum_{p=1}^{\infty} \min(\lambda_n^2 \theta_p, \Delta_n).$$