Change Point Estimation by Local Linear Smoothing under a Weak Dependence Condition

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We consider a change point problem in regression estimation. Observations (X_i, Y_i), $i = 1, \ldots, n$ are governed by the model $Y_i = m(X_i) + \sigma(X_i)\varepsilon_i$, where $(\varepsilon_i)_{i\in\mathbb{Z}}$ is independent and identically distributed, and independent of $(X_i)_{i\in\mathbb{Z}}$. The latter sequence satisfies a weak dependence condition proposed by Dedecker and Prieur [4]. We essentially study the basic situation where the regression function has a unique change point. The construction of the jump estimate process, $t \to \hat{\gamma}(t)$, is based on local linear regression. Under a positivity condition regarding the asymmetric kernel involved, we prove the convergence to a compound Poisson process with an additional drift, of a local dilated-rescaled version of $\hat{\gamma}(t)$. We also derive asymptotic normality results.

Key words and phrases. Central limit theorem, change point, infinitely divisible distributions, Lindeberg theorem, local linear regression, nonparametric regression, stationary sequences, weakly dependent sequences.

1 Introduction

This paper deals with change point problems in regression estimation. Number of authors in the recent years have studied such problems in a parametric or non parametric setting. Among the applications we can cite biostatistics, signal processing, econometrics and so on. In the case of abrupt changes example given for prices, wages, ..., it is of main interest to test for the existence of change point but also to estimate the location and the size of the jumps. The setting chosen in this paper is the non parametric one. We refer to the monograph edited by Müller *et al.* [13] for references on parametric methods. We consider the following regression model : $Y_i = m(X_i) + \sigma(X_i)\varepsilon_i\,,\ i = 1, \ldots, n$, where $(X_i)_{i\in\mathbb{Z}}$ is a stationary and weak dependent sequence. $(\varepsilon_i)_{i\in\mathbb{Z}}$ is independent and identically distributed, with mean zero and variance unity, and is independent of $(X_i)_{i\in\mathbb{Z}}$. The regression function $m(.)$ is smooth but for some points where jumps in the function itself or in one of its derivatives occur. We are interested in estimating the size and the location of the jumps. Such a model has been studied in the case where the observations (X_i, Y_i)'s are independent and identically distributed.

Several approaches were proposed in this framework. Of course, we do not pretend to give a complete account of the literature on the topic but rather insist on methods based on differences between left and right estimates. This approach consists in estimating the right (resp. left) limit $m_+(\tau)$ (resp. $m_-(\tau)$) using data located at the right (resp. left) of point τ. An estimate for a possible jump $\gamma(\tau) = m_+(\tau) - m_-(\tau)$ follows then. Let us cite for example [14, 12, 18, 10]. Müller [14] uses kernel smoothing with left kernel K_+ with support in $[-1, 0]$ and K_- defined by $K_-(x) = K_+(-x)$. He obtains, for the estimate of a change point, the convergence rate $n^{-(1+\varepsilon)}$ for some $\varepsilon > 0$. Wu and Chu [18] give then improvements, still using kernel smoothing. Grégoire and Hamrouni [10] appeal to linear local regression. Their essential motivation is that this method has no edge effects contrarily to the kernel one. The achieved rate of convergence for the bias is the same near the boundaries as inside the interval. Loader's paper [12] is based on the same idea, but in a rather different setting. Loader [12] uses indeed a fixed regular design and assume the noise to be gaussian with constant variance, while Grégoire and Hamrouni [10] work with a random design, without any particular assumption on the noise distribution. In their model, the variance is allowed to depend on the location. In this paper, we follow their approach. However, in many cases, physical constraints entail that serious modeling cannot be done only using independent sequences. It justifies our choice to study what happens in a dependent frame. In our model, the sequence $(X_i)_{i \in \mathbb{Z}}$ is no longer independent. We assume that this sequence is ϕ-dependent in the sense of Dedecker and Prieur [4]. Exact definitions and properties of ϕ-dependent sequences are recalled in Section 2. This notion of dependence, contrarily to classical mixing conditions (see for example the monographs by Doukhan [5], Rio [17] and Bradley [2]), covers many commonly used classes of models (see examples in Section 2.1).

In this paper we essentially focus on the situation where there is a unique change point τ in the regression function itself. However, the method can be extended to the case of more than one jump, or to the case where the change point is in a derivative [10]. The case where the location of τ is known has already been studied in a weak dependence setting by Ango Nze and Prieur [1]. This case corresponds to some practical situations where, for example, an experimenter produces himself a change point (beginning of a new treatment, surgical operation, on so on). The last authors worked with dependent sequences in the sense of Doukhan and Louhichi [6], and did not give any results concerning the case where the jump location τ is unknown. Working under ϕ-dependence, we recover the central limit theorem of Ango Nze and Prieur [1]. As far as we know, there is no previous results dealing with dependent data in the case where the jump location is unknown. In this paper, we prove convergence results for the jump estimate. Denote $t \to \hat{\gamma}(t) = \hat{m}_+(t) - \hat{m}_-(t)$ our estimated jump process, where $\hat{m}_+(t)$ and $\hat{m}_-(t)$ are the estimates of the left and right limits of $m(.)$ at point t, obtained by linear local regression with positive kernel $K_+(.)$ supported on $[-1, 0]$, and K_- defined by $K_-(x) = K_+(-x)$. Let us assume in this paper (for sake of simplicity) that $\gamma(\tau) > 0$. In this case, we define $\hat{\tau}$ as a point where $\hat{\gamma}(.)$ is maximum. We assume that $K_+(0) > 0$. It makes the samples of the process $\hat{\gamma}(t)$ to be discontinuous, but it allows us to estimate the jump location τ with rate n^{-1}.

In the case where τ is known, the main tool to obtain the central limit theorem is a Lindeberg adaptation in a weak dependence frame [15, 1]. To study the location estimate, we need the deviation process

$$\mathcal{Z}_n(z) = \alpha(n, h_n) \left(\hat{\gamma}(\tau + \frac{h_n}{\beta(n, h_n)} z) - \hat{\gamma}(\tau) \right),$$

where the bandwidth h_n is a sequence of positive real numbers which tends to zero as n tends to infinity. The location estimate we have chosen $\hat{\tau}$ satisfies $\hat{\tau} = \arg\sup \mathcal{Z}_n(z)$ as z lies in $[-M, M]$ for some large enough M. We get in that case that, when the rescaling and dilating parameters $\alpha(n, h_n)$ and $\beta(n, h_n)$ are chosen in a convenient way, $\mathcal{Z}_n(z)$ converges to a compound Poisson process with an additional drift, which yields the consistency of $\hat{\tau}$. We clarify in this paper that the limiting process is the same as in the independent frame. This is a new result, which is important as it makes clear that even in the case where the observations are no longer independent, we can estimate a change point in the regression function. To prove it, we adapt results of Dedecker and Louhichi [3] on convergence of partial sums of an array with stationary rows to infinitely divisible distributions. Their conditions are written in terms of conditional expectations, which can be checked in our setting.

The paper is organized as follows. Section 2 is devoted to the dependence frame. In Section 3, we give the asymptotic results concerning the case where the jump location is known. In Section 4, we write our main result in the case where the jump location is unknown. We also give several corollaries concerning asymptotic normality. The proofs of the results of Section 4 are written in Section 5, except for some technical points whose proofs are postponed to Appendix A.

2 Dependence setting

In our regression model $Y_i = m(X_i) + \sigma(X_i)\varepsilon_i$, the sequence $(\varepsilon_i)_{i\in\mathbb{Z}}$ is independent and identically distributed, and independent of $(X_i)_{i\in\mathbb{Z}}$. The sequence $(X_i)_{i\in\mathbb{Z}}$ is assumed to be weakly dependent in a sense we precise below.

2.1 ϕ-dependence

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Let X be a real-valued random variable and $\mathcal{M}$ a σ-algebra of $\mathcal{A}$. We then define

$$\phi(\mathcal{M}, X) \quad = \quad \sup_{t\in\mathbb{R}} \|F_{X|\mathcal{M}}(t) - F_X(t)\|_\infty \,.$$

This coefficient has been first introduced by Dedecker and Prieur [4]. It is a weak version of the classical ϕ-mixing coefficient introduced by Ibragimov [11]. It measures the gap between the conditional distribution function of X given $\mathcal{M}$ and the distribution function of X. Its main advantage is that it is easier to compute. Before going further, let us define the coefficients $\phi(i)$ of a sequence of stationary real-valued random variables. Let us consider $(X_i)_{i\in\mathbb{Z}}$, a stationary sequence of integrable real valued random variables. For all $i \in \mathbb{Z}$, define $\mathcal{M}_i := \sigma(X_l, l \leq i)$. The sequence of coefficients $\phi(i)$ is then defined by

$$\phi(i) = \phi(\mathcal{M}_0, X_i)\,.$$

We say that the sequence $(X_i)_{i\in\mathbb{Z}}$ is ϕ-dependent if $\phi(n) \xrightarrow[n\to+\infty]{} 0$.

Examples Many classical models can be proved to be ϕ-dependent. Let us cite the three following examples :

- *Causal functions of stationary sequences* Let $(\xi_i)_{i\in\mathbb{Z}}$ be a stationary sequence of random variables with values in a measurable space $\mathcal{X}$. Assume that there exists a function H defined on a subset of $\mathcal{X}^{\mathbb{Z}}$, with values in $\mathbb{R}$ and such that $H(\xi_0, \xi_{-1}, \xi_{-2}, \ldots)$ is defined almost surely. Then $(X_n)_{n\in\mathbb{Z}}$ defined for all n in $\mathbb{Z}$ by

  $$X_n = H(\xi_n, \xi_{n-1}, \xi_{n-2}, \ldots)$$

  is called a causal function of $(\xi_i)_{i\in\mathbb{Z}}$. A particular case is the example of causal linear processes defined for all n in $\mathbb{Z}$ by

  $$X_n = \sum_{j\geq 0} a_j \xi_{n-j}\,.$$

- *Iterated random functions* Let $(X_n)_{n\geq 0}$ be a real-valued stationary Markov chain, such that $X_n = F(X_{n-1}, \xi_n)$ for some measurable function F and some i.i.d. sequence $(\xi_i)_{i>0}$ independent of X_0.

- *Dynamical systems on* $[0, 1]$ Let $I = [0, 1]$, T be a map from I to I and μ a probability measure invariant by T. Then $(X_n)_{n\geq 0}$ is defined by $X_i = T^i$. Then $(X_i)_{i\geq 0}$ is a strictly stationary sequence of random variables from (I, μ) to I.

For each of these models, explicit bounds for the coefficient ϕ are computed under appropriate assumptions in Dedecker and Prieur [4]. In particular we get conditions under which $\sum_{i=1}^{+\infty} \phi(i)$ is finite.

2.2 A covariance inequality

In this section, we recall a very usefull covariance inequality.

Proposition 2.1 (Dedecker and Prieur [4]) *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Let X and Y be two real-valued random variables and h be a σ-BV function. Assume that Y, $h(X)$ and $Yh(X)$ are integrable. Let $\mathcal{M}$ be a σ-algebra of $\mathcal{A}$. If Y is $\mathcal{M}$-measurable, we have the inequality*

$$|\mathrm{Cov}(Y, h(X))| \leq \|dh\| \, \|Y\|_1 \, \phi(\mathcal{M}, X) \, .$$

This proposition allows to get useful bounds to prove the results of Sections 3 and 4.

3 Asymptotic results when the jump location is known

In this section, we are interested in the case where the location τ of the discontinuity is known. We first precise assumptions on the model and then give asymptotic results for the estimate of the jump's size.

3.1 Assumptions on the model

The sequence of observations $(X_n, Y_n)_{n\in\mathbb{Z}}$ is a stationary sequence of random variables valued in $\mathbb{R}^2$. We have the following model: $Y_i = m(X_i) + \sigma(X_i)\varepsilon_i$ where the function $m(\cdot)$ is smooth except in the known point τ, the function $\sigma(\cdot)$ is continuously differentiable, and the real valued sequence of random variables $(\varepsilon_n)_{n\in\mathbb{Z}}$ is a centered and normalized independent and identically distributed sequence. We assume moreover that $(X_n)_{n\in\mathbb{Z}}$ and $(\varepsilon_n)_{n\in\mathbb{Z}}$ are mutually independent.

In the following we will make use of the additional assumptions:

1. $\mathbb{E}\,\varepsilon_0 = 0$, $\mathbb{E}\,(\varepsilon_0^2) = 1$, $\mathbb{E}\,|\varepsilon_0|^3 < \infty$.
2. X_0 takes values in $[0, 1]$. It has a density f strictly positive and continuous.
3. The regression function $m(\cdot) = \mathbb{E}(Y_0|X_0 = \cdot)$ is two times continuously differentiable in each of the intervals $[0, \tau]$ and $[\tau, 1]$. In particular, m and its two first derivatives are left and right limited at point τ.
4. The conditional variance $\sigma^2(\cdot) = \mathrm{Var}\,(Y_0|X_0 = \cdot)$ is continuously differentiable.
5. The change point τ of $m(\cdot)$ lies in $]0, 1[$.

3.2 Local linear estimators

The functions m and σ are unknown. We use kernel estimators to estimate these functions. The definitions below are quite similar to those introduced by Fan and Gijbels [8]. Let K be a kernel, and $h_n = h$ a window. The real parameters $\hat{\alpha}_x$ and $\hat{\beta}_x$ are obtained by solving the following minimizing problem $\sum_{i=1}^{n}(Y_i - \alpha_x - \beta_x(x - X_i))^2 K_i(x)$, with $K_i(x) = K\left(\frac{x - X_i}{h}\right)$.

Definition 3.1 *The local linear estimator of $m(\cdot)$ is*

$$\widehat{m}(x) = \hat{\alpha}_x = \frac{\sum_{i=1}^{n} \omega_i(x) Y_i}{\sum_{i=1}^{n} \omega_i(x)},$$

with $\omega_i(x) = K_i(x)\left(S_2(x) - (x - X_i)S_1(x)\right)$, where, for $l \in \mathbb{N}$, $S_l(x) = \sum_{i=1}^{n}(x - X_i)^l K_i(x)$.

3.3 Right and left estimators

To make this article more readable, we make here the following assumption. We consider a kernel $K_+ : [-1, 0] \to \mathbb{R}_+$ continuously differentiable. We define then:

$$\begin{array}{rl} K_l^+ = \int_{-1}^{0} x^l K_+(x)dx, & \text{and} \quad L_l^+ = \int_{-1}^{0} x^l K_+^2(x)dx, \; l \in \mathbb{N}, \\ B_+ = (K_2^+)^2 - K_3^+ K_1^+, & \text{and} \quad V_+ = \int_{-1}^{0}(K_2^+ - xK_1^+)^2 K_+^2(x)dx. \end{array}$$

We also define $K_-(x) = K_+(-x)$, and the quantities K_l^-, L_l^-, B_- and V_- relatively to K_-. We have then $V_+ = V_-$, $B_+ = B_-$ and $V_+ = (K_2^+)^2 L_0^+ + (K_1^+)^2 L_2^+ - 2K_1^+ K_2^+ L_1^+$.

Moreover, there is no loss of generality to querry for the normalization $K_2 K_0 - K_1^2 = 1$. The kernels K_+ and K_- allow then the definition of the local linear estimators $\widehat{m}_+$ and $\widehat{m}_-$ defined by Definition 3.1.

3.4 Results

We suppose satisfied assumptions of Section 3.1. Let us state the main result of the present section (Theorem 3.1) and a straightforward corollary (Corollary 3.1).

Theorem 3.1 *Assume that the sequence $(X_i)_{i \in \mathbb{Z}}$ is ϕ-dependent with $\sum_{i=1}^{+\infty} \phi(i) < +\infty$. If $nh^5 \to 0$, $nh \to +\infty$ then*

$$\sqrt{nh}\left(\widehat{m}_-(\tau) - m_-(\tau), \widehat{m}_+(\tau) - m_+(\tau)\right)'$$

converges to the two-dimensional normal law :

$$\mathcal{N}\left(0, \left(V_+ \sigma^2(\tau)/f(\tau)\right) I_2\right)$$

where I_2 is the identity matrix.

Corollary 3.1 *Let assume the sequence $(X_i)_{i \in \mathbb{Z}}$ to be ϕ-dependent with $\sum_{i=1}^{+\infty} \phi(i) < +\infty$. If $nh^5 \to 0$, $nh \to +\infty$ then* $\sqrt{nh}\left(\widehat{\gamma}(\tau) - \gamma(\tau)\right) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, 2\frac{\sigma^2(\tau)}{f(\tau)} V_+\right)$.

The proof of Theorem 3.1 is a variation on Lindeberg-Rio's techniques [16]. These results have been proved by Ango Nze and Prieur [1] under the condition that the sequence $(X_i)_{i\in\mathbb{Z}}$ is s-dependent in the sense of Doukhan and Louhichi [6]. This condition is most of the time weaker than the ϕ one (see Proposition 2 in Dedecker and Prieur, [4] for comparison of the coefficients). The proofs are written in [15]. The proofs of Theorem 3.1 and Corollary 3.1 follow essentially the same lines, except for the control of the covariance terms, for which we use the covariance inequality of Proposition 2.1. They are therefore not detailed in this paper. Working with the ϕ-dependence assumption allows us to study the case where the change point location is unknown, using the covariance inequality of Proposition 2.1. Hence we focus in the following on the ϕ-dependence setting.

4 Case where the jump location is unknown

In this section, we follow the approach of Grégoire and Hamrouni [10]. Our result is new as it is stated for dependent data.

4.1 Introduction

Recall that we define, for all $t \in [0,1]$, $\hat{\gamma}(t) = \hat{m}_+(t) - \hat{m}_-(t)$ and that for sake of simplicity, we assume that $\gamma(\tau) > 0$. We introduce the following natural estimate of τ

$$\hat{\tau} = \inf\{t \in \kappa\,;\ \hat{\gamma}(t) = \sup_{x\in\kappa} \hat{\gamma}(x)\},$$

where κ is a compact included in $]0,1[$.

To study the asymptotic distributions of $\hat{\tau}$ and $\hat{\gamma}(\hat{\tau})$, we introduce the following rescaled-dilated version of the process $\hat{\gamma}(t)$ around τ

$$\mathcal{Z}_n(z) = \alpha(n,h_n)\left(\hat{\gamma}(\tau + \frac{h_n}{\beta(n,h_n)}z) - \hat{\gamma}(\tau)\right), \quad z \in [-M,M].$$

Eddy [7], to estimate the mode of a distribution, is one of the firsts to have used such arguments. If M is large enough, we have $\hat{\tau} = \tau + \frac{h_n}{\beta(n,h_n)}\hat{z}$, where $\hat{z} = \arg\sup_{z\in[-M,M]} \mathcal{Z}_n(z)$. Then, choosing properly $\alpha(n,h_n)$ and $\beta(n,h_n)$, we prove that $\mathcal{Z}_n(z)$ converges to a process $\mathcal{Z}$. So $z_{\max}(n) = \arg\sup_z \mathcal{Z}_n(z)$ converges to $z_{\max} = \arg\sup_z \mathcal{Z}(z)$. We also deduce from these convergences informations on the rate of convergence of $\hat{\tau} = \tau + \frac{h_n}{\beta(n,h_n)} z_{\max}(n)$.

4.2 Assumptions

- $\mathbf{H_1}$. $\lim_{n\to\infty} h_n = 0$ and $\lim_{n\to\infty} nh_n = \infty$.
- $\mathbf{H_2}$. $\lim_{n\to\infty} \frac{h_n}{\beta(n,h_n)} = 0$, $\lim_{n\to\infty} \alpha(n,h_n) = \infty$ and $\lim_{n\to\infty} \beta(n,h_n) = \infty$.

- $\mathbf{H_3}$.
  - **0**. $K_+(0) > 0$.
  - **00**. $K_+(-1) = 0$.
  - **1**. $0 < \lim_{n\to\infty} \frac{\alpha(n,h_n)}{\beta(n,h_n)} = L_4 < \infty$.
  - **2**. $\lim_{n\to\infty} \frac{\alpha(n,h_n)}{nh_n} = L_5 < \infty$.

If $H3$ is satisfied, we define

$$\lim_{n\to\infty} \frac{\alpha^2(n,h_n)}{nh_n\beta(n,h_n)} = L_4 \times L_5 = L_6 < \infty\,.$$

Assumption $H3.00$ is unnecessary, but it is set for sake of simplicity in the calculations.

4.3 Notations

We use the following notations :

- $\lambda_1 = L_5\, M_+(0)$,
- if $L_5 > 0$, $\lambda_2 = \frac{L_6}{L_5^2} f_X(\tau)$,
- $\lambda_3 = L_4 M_+(0) f_X(\tau) = \lambda_1\lambda_2$.

Let $y := \frac{z}{\beta(n,h_n)}$. Let $M^{\pm}(x) := (K_2^{\pm} - xK_1^{\pm})K_{\pm}(x)$. We define for any z,

- $\varphi^{\pm}_{n,z}(t) := \frac{\alpha(n,h_n)}{nh_n}\left(M^{\pm}\left(\frac{\tau-t}{h_n}+y\right) - M^{\pm}\left(\frac{\tau-t}{h_n}\right)\right)(m(t)-m^{\pm}(\tau))$,
- $\varphi_{n,z}(t) := \varphi^{+}_{n,z}(t) - \varphi^{-}_{n,z}(t)$,
- $\tilde{\varphi}_{n,z}(t) := \varphi_{n,z}(t) - \mathbb{E}\left(\varphi_{n,z}(X_1)\right)$,
- $\psi^{\pm}_{n,z}(t) := \frac{\alpha(n,h_n)}{nh_n}\left(M^{\pm}\left(\frac{\tau-t}{h_n}+y\right) - M^{\pm}\left(\frac{\tau-t}{h_n}\right)\right)\sigma(t)$,
- $\psi_{n,z}(t) := \psi^{+}_{n,z}(t) - \psi^{-}_{n,z}(t)$, and
- for any i in $\mathbb{N}$, $\tilde{U}_{i,n,z} := |\tilde{\varphi}_{n,z}(X_i)| + |\psi_{n,z}(X_i)|$.

4.4 Main result

We can now write the key result of this paper.

Theorem 4.1 *Assume that H_1, H_2 and H_3 are satisfied. Assume that $(X_i)_{i\in\mathbb{Z}}$ is ϕ-dependent with $\sum_{i=1}^{+\infty}\phi_i < +\infty$. We assume moreover that for any fixed z_1, z_2, for any fixed $k \in \mathbb{N}^*$,*

$$n\,\mathrm{Cov}\left(\tilde{U}_{0,n,z_1}, \tilde{U}_{k,n,z_2}\right) \xrightarrow[n\to+\infty]{} 0\,. \tag{1}$$

Then we have

$$\mathcal{Z}_n \Rightarrow \mathcal{Z}\,, \qquad \text{on} \quad \mathcal{D}\left([-M, M]\right) \tag{2}$$

where

$$\mathcal{Z}(z) = \frac{\lambda_3}{f_X(\tau)}\gamma(\tau)|z| + \frac{\lambda_1}{f_X(\tau)}\mathcal{N}(z)\,, \tag{3}$$

with $\mathcal{N}(z)$ defined by

$$\mathcal{N}(z) = \begin{cases} \sum_{i=1} N_z^+(-\gamma(\tau) - 2\sigma(\tau)\varepsilon_i^+) & \text{if } z \geq 0\,, \\ \sum_{i=1} N_{-z}^-(-\gamma(\tau) + 2\sigma(\tau)\varepsilon_i^-) & \text{if } z < 0\,. \end{cases} \tag{4}$$

The sequences (ε_i^+) and (ε_i^-) are independent and built with i.i.d. variables distributed as the model error variable ε. N_z^+ and N_z^- are independent homogeneous Poisson processes with λ_2 as parameter, and are independent of the sequences (ε_i^+) and (ε_i^-).

Remark 1 *The limit process $\mathcal{Z}$ is a bilateral compound Poisson process with an additional drift. Under the alternative assumption $\gamma(\tau) < 0$, we would have changed the sign before $2\sigma(\tau)\varepsilon_i^{\pm}$ in both cases $z \geq 0$ and $z < 0$. Note that the limit process $\mathcal{Z}$ is the same as in the independent case. We prove indeed, using mainly covariance inequality of Proposition 2.1 and the ϕ-dependence property of the sequence $(X_i)_{i\in\mathbb{Z}}$, that the covariance terms do not affect the limit.*

Remark 2 *Theorem 4.1 applies to the models of Section 2.3, as soon as condition (1) is satisfied. Let us precise what happens for two classes of examples.*

Claim 1
Assume that there exists joint densities for the couples (X_0, X_k), $f_k(x, y)$, uniformly bounded with respect to k :

$$\sup_{k\geq 1}\sup_{x,\,y}|f_k(x, y)| < \infty\,. \tag{5}$$

Then assumption (1) is satisfied.

Proof of Claim 1
We have in that case

$$n\,\mathrm{Cov}\left(\tilde{U}_{i,n,z_1}, \tilde{U}_{0,n,z_2}\right) = \mathcal{O}\left(n\left(\frac{\alpha(n,h_n)}{nh_n}\right)^2\left(\frac{h_n}{\beta(n,h_n)}\right)^2\right) = \mathcal{O}\left(\frac{1}{n}\right)\,,$$

which tends to 0 *as* n *tends to infinity.* □

Claim 2
Let (T,μ) *be a dynamical on* $[0,1]$. *Let* X_0 *follow the invariant law* μ *and for any* k *in* $\mathbb{N}^*$ *define* $X_k = T^k X_0$. *Assume that* μ *is absolutely continuous with respect to Lebesgue. For the sequence* $(X_i)_{i\in\mathbb{N}}$ *assumption (1) is satisfied.*

Proof of Claim 2
We have in that case

$$n\,\mathrm{Cov}\left(\tilde{U}_{i,n,z_1},\tilde{U}_{0,n,z_2}\right)=\mathcal{O}\left(n\left(\frac{\alpha(n,h_n)}{nh_n}\right)^2\left(\frac{1}{\beta(n,h_n)}\right)^2\frac{1}{h_n}\right)=\mathcal{O}\left(\frac{1}{nh_n}\right),$$

which tends to 0 *as* n *tends to infinity.* □

Let us give now straightforward corollaries of Theorem 4.1.

Corollary 4.1 *Assume that* H_1, H_2, $H_{3,0}$ *and* $H_{3,00}$ *are satisfied. Assume that* $(X_i)_{i\in\mathbb{Z}}$ *is* ϕ*-dependent with* $\sum_{i=1}^{+\infty}\phi(i)<+\infty$, *and that assumption (1) is satisfied. Then*

$$\frac{\beta(n,h_n)}{h_n}(\hat{\tau}-\tau)\xrightarrow{\mathcal{D}}\mathcal{T},$$

where $\mathcal{T}$ *is a* $\mathbb{R}$*-valued variable defined as*

$$\mathcal{T}=argsup_z\left\{-\frac{\lambda_3}{f_X(\tau)}\gamma(\tau)|z|+\frac{\lambda_1}{f_X(\tau)}\mathcal{N}(z)\right\}.$$

Corollary 4.1 is deduced from Theorem 4.1. The proof is the same as in the independent case and is written in Section 3 of Grégoire and Hamrouni [10]. If we take $\beta(n,h_n)=nh_n$ in Corollary 4.1, we reach the rate n^{-1} as far as the sequence of bandwiths h_n satisfies $nh_n\to+\infty$. We derive now asymptotic normality for $\hat{\tau}$ and $\gamma(\hat{\tau})$.

Corollary 4.2 *Assume that* H_1 *is satisfied and that* $nh_n^5\to 0$. *Assume that* $H_{3,0}$ *and* $H_{3,00}$ *are satisfied. Assume that* $(X_i)_{i\in\mathbb{Z}}$ *is* ϕ*-dependent with* $\sum_{i=1}^{+\infty}\phi(i)<+\infty$, *and that assumption (1) is satisfied. Then*

$$\sqrt{nh_n}(\hat{\gamma}(\hat{\tau})-\gamma(\tau))\xrightarrow{\mathcal{D}}\mathcal{N}\left(0,\frac{2\sigma^2(\tau)}{f_X(\tau)}V_+^2\right).$$

The proof can be easily deduced from Corollary 3.1 and Theorem 4.1.

Corollary 4.3 *Let $\hat{f}_X(.)$ and $\sigma^2(.)$ be consistent estimators of $f(.)$ and $\sigma^2(.)$ such as* $\sup_{t\in(0,1)} |\hat{f}_X(t) - f_X(t)| = o_P(1)$ *and* $\sup_{t\in(0,1)} |\hat{\sigma}^2(t) - \sigma^2(t)| = o_P(1)$.

The estimator $\hat{f}_X(.)$ is supposed to be nonnegative. Under the assumptions of Corollary 4.2, we have

$$\sqrt{nh_n}\frac{\sqrt{\hat{f}_X(\hat{\tau})}(\hat{\gamma}(\hat{\tau}) - \gamma(\tau))}{(2\hat{\sigma}^2\hat{\tau}))^{1/2}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, V_+)\,.$$

We refer to Grégoire and Hamrouni [10] for the choice of convenient estimators.

4.5 Key argument in the proof

We define $\mathcal{M}_n^{\pm}(z)$ by

$$\mathcal{M}_n^{\pm}(z) = \sum_{i=1}^{n} \left(\varphi_n^{\pm}(X_i) + \psi_n^{\pm}(X_i)\varepsilon_i\right)\,.$$

To prove Theorem 4.1, we prove the same result on $\mathcal{M}_n(z) := \mathcal{M}_n^{+}(z) - \mathcal{M}_n^{-}(z)$ which is asymptotically equivalent to $\mathcal{Z}_n(z)$.

Theorem 4.2 *Assume that the sequence $(X_i)_{i\in\mathbb{Z}}$ is ϕ-dependent with $\sum_{i=1}^{+\infty}\phi(i) < +\infty$. Then, under H_1 and H_3,*

$$\begin{aligned}\mathcal{Z}_n(z) &= \tfrac{1+o_P(1)}{f(\tau)}(\mathcal{M}_n^{+}(z) - \mathcal{M}_n^{-}(z)) + o_P(1)\\ &= \tfrac{1+o_P(1)}{f(\tau)}\mathcal{M}_n(z) + o_P(1)\,.\end{aligned}$$

The proof of Theorem 4.2, which is rather technical and lengthy, is postponed to Appendix A. The proof of the convergence of $\mathcal{M}_n(z)$ relies on the fact that we can write $\mathcal{M}_n(z)$ as the sum of the terms of a row in a triangular array. The proof is written in Section 5 below.

5 Proof of Theorem 4.1

In order to prove Theorem 4.1, we proceed in two steps. First we prove the 2-dimensional convergence of the process $\mathcal{M}_n(z)$, then we prove that the process $(\mathcal{M}_n(z), z \in [-M, M])$ is tight in $\mathcal{D}([-M, M])$.

5.1 2-dimensional convergence

To prove the 2-dimensional convergence, according to Cramer-Wold device, we need to show that for any (a, b) and for any pair $(z_1, z_2) \in [-M, M] \times (-M, M]$, $a\mathcal{M}_n(z_1) + b\mathcal{M}_n(z_2)$ has an asymptotic distribution. Define $\varphi_n(t) := a\varphi_{n,z_1}(t)+b\varphi_{n,z_2}(t)$. $\tilde{\varphi}_n$ and $\psi_n(t)$ are defined similarly. Using the notations of Section 4.3, define $T_{i,n} := \varphi_n(X_i)+\psi_n(X_i)\varepsilon_i$ and $\tilde{U}_{i,n} := a\tilde{U}_{i,n,z_1}+b\tilde{U}_{i,n,z_2}$. We can write $a\mathcal{M}_n(z_1)+b\mathcal{M}_n(z_2) = \sum_{i=1}^{n} T_{i,n}$. Hence, we have to prove the convergence of the cumulative sum of the terms of the same row of a triangular array, $\sum_{i=1}^{n} T_{i,n}$. In the independent case, the proof of Grégoire and Hamrouni [10] relies on the classic result of convergence of Gnedenko and Kolmogorov [9] to infinitely divisible distributions. Here we can not apply this result. However, Dedecker and Louhichi [3] have extended this result under a condition which can be expressed in terms of conditional expectations. In the following, we prove that we can apply their result. Define $\tilde{T}_{k,n} := T_{k,n} - \mathbb{E}T_{k,n} = \tilde{\varphi}_n(X_k) + \psi_n(X_k)\varepsilon_k$. For any $k \in \mathbb{N}$, define $\mathcal{M}_k := \sigma(X_j, j \leq k)$. Define $S_n(t) := \tilde{T}_{1,n} + \cdots + \tilde{T}_{[nt],n}$, for any $t \in [0,1]$. We are interested in the convergence in distribution of $S_n(1)$. Let $\mathcal{H}$ be the space of continuous real functions φ such that $x \to |(1+x^2)^{-1}\varphi(x)|$ is bounded. If F is any distribution function, define also μ_F^1 as the probability measure with characteristic function

$$\phi_F(z) = exp\left(\int (e^{izx} - 1 - izx)\frac{1}{x^2}dF(x)\right) \tag{6}$$

and $\mu_F^t := \mu_{tF}^1$. In order to prove the convergence in distribution of $S_n(1)$, we will prove the stronger result $\mathbf{S_1}$, that is we will prove that there exists some distribution function Γ such that for any φ in $\mathcal{H}$, any t in $[0,1]$ and any positive integer k,

$$\mathbf{S_1}(\varphi) \quad \lim_{n\to+\infty} \|\mathbb{E}\left(\varphi(S_n(t)) - \mu_\Gamma^t(\varphi)|\mathcal{M}_k\right)\|_1 = 0\,.$$

The main tool of the proof is Corollary 3 of Dedecker and Louhichi [3] recalled further. Let us first introduce some notations.

Definition 5.1 *For any positive integer N, define*

$$R_1(N,T) = \lim_{t\to 0}\limsup_{n\to+\infty}\sup_{N\leq m\leq [nt]} n\|\tilde{T}_{0,n}\sum_{k=N}^{m}\mathbb{E}(\tilde{T}_{k,n}|\mathcal{M}_0)\|_1\,,$$

and $N_1(T) = \inf\{N > 0 : R_1(N) = 0\}$ ($N_1(T)$ may be infinite). We say that $(\tilde{T}_{i,n})$ satisfies the weak-dependence condition **WD** *if*

$$\lim_{t\to 0}\limsup_{n\to+\infty}\frac{1}{t}\|\mathbb{E}(S_n(t)|\mathcal{M}_0)\|_1 = 0 \tag{7}$$

and $R_1(N,T)$ tends to zero as N tends to infinity.

WD is a weak dependence condition. In addition to it, we need to control some residual terms. Condition (8) is a kind of equiintegrability condition.

Definition 5.2 *For any (k,n) in $(\mathbb{N}\times\mathbb{Z})$, define $S_{k,n}$ by $S_{k,n} = 0$ if $k \leq 0$ and $S_{k,n} = \tilde{T}_{1,n} + \cdots + \tilde{T}_{k,n}$ otherwise. For any positive integer N define*

$$R_2(N,T) = \limsup_{t\to 0}\limsup_{n\to+\infty}\frac{1}{t}\mathbb{E}\left(\sum_{k=1}^{[nt]}(\tilde{T}_{k,n}^2 + 2|\tilde{T}_{k,n}(S_{k-1,n} - S_{k-N,n})|)(1\wedge|S_{k-N}|)\right)\,, \tag{8}$$

and $N_2(T) = \inf\{N > 0 : R_2(N) = 0\}$ ($N_2(T)$ may be infinite). We say that the array $(\tilde{T}_{i,n})$ is **EQ** *if $n\mathbb{E}(\tilde{T}_{0,n}^2)$ is bounded and if $R_2(N,T)$ tends to zero as N tends to infinity.*

We define $N_0(T) := N_1(T) \vee N_2(T)$. We can now give the following result which is a weak version of [3, Corollary 3, page 7].

Corollary 5.1 *Assume that $(\tilde{T}_{i,n})$ is* **WD** *and* **EQ** *with $N_0(T) = 1$. Assume moreover that $\mathbb{E}\tilde{T}_{0,n}^2 \xrightarrow[n\to+\infty]{} 0$. Then $\mathbf{S_1}$ holds for a distribution function Γ if and only if : for any continuity point x (including $+\infty$) of the function $x \to \Gamma(x)$,*

$$\lim_{t\to 0}\limsup_{n\to+\infty}\left\|\mathbb{E}\left(\frac{1}{t}\sum_{k=1}^{[nt]}\tilde{T}_{k,n}^2\mathbb{1}_{\tilde{T}_{k,n}\le x} - \Gamma(x)|\mathcal{M}_0\right)\right\|_1 = 0\,. \tag{9}$$

In [3, Corollary 3], you can consider a random distribution function instead of Γ. We refer to their paper for further details. In order to apply Corollary 5.1, we have to prove that the assumptions are satisfied. In the following, C is some constant which may vary from line to line. Moreover, recall that for h_n, $\alpha(n, h_n)$ and $\beta(n, h_n)$ we assume H_1 to H_3.

<u>Proof of $N_1(T) = 1$:</u> We want to prove

$$\lim_{t\to 0}\limsup_{n\to+\infty}\sup_{1\le m\le[nt]} n\left\|\tilde{T}_{0,n}\sum_{k=1}^{m}\mathbb{E}(\tilde{T}_{k,n}|\mathcal{M}_0)\right\|_1 = 0\,. \tag{10}$$

On one hand, using the ϕ-dependence property of $(X_n)_{n\in\mathbb{Z}}$, we get

$$\begin{aligned} n\,\mathbb{E}\left|\tilde{T}_{0,n}\mathbb{E}(\tilde{T}_{k,n}|\mathcal{M}_0)\right| &\le n\,\|\mathbb{E}(\tilde{\varphi}_n(X_k)|\mathcal{M}_0)\|_\infty\,\mathbb{E}|\tilde{T}_{0,n}| \\ &= \mathcal{O}\left(n\left(\frac{\alpha(n,h_n)}{nh_n}\right)^2\frac{h_n}{\beta(n,h_n)}\phi(k)\right) \le C\,\phi(k)\,, \end{aligned} \tag{11}$$

for some constant C. We have indeed $\|d\tilde{\varphi}_n\| = \mathcal{O}\left(\frac{\alpha(n,h_n)}{nh_n}\right)$.

On the other hand we have, for any fixed k in $\mathbb{N}^*$,

$$n\,\mathbb{E}\left|\tilde{T}_{0,n}\mathbb{E}(\tilde{T}_{k,n}|\mathcal{M}_0)\right| \le n\,\mathbb{E}\left(\tilde{U}_{0,n}\tilde{U}_{k,n}\right) = n\,\mathrm{Cov}\left(\tilde{U}_{0,n},\tilde{U}_{k,n}\right) + n\,\left(\mathbb{E}\tilde{U}_{0,n}\right)^2\,, \tag{12}$$

as the sequence $(X_i)_{i\in\mathbb{Z}}$ is stationary. Now as $\mathbb{E}\tilde{U}_{0,n} = \mathcal{O}\left(\frac{h_n}{\beta(n,h_n)}\right)$, using assumption (1), we get that the right hand term in (12) tends to 0 as n tends to infinity.

Hence, as $\sum_{k\ge 1}\phi_k < +\infty$, the dominated convergence theorem yields for any t

$$\sup_{1\le m\le[nt]} n\left\|\tilde{T}_{0,n}\sum_{k=1}^{m}\mathbb{E}(\tilde{T}_{k,n}|\mathcal{M}_0)\right\|_1 \xrightarrow[n\to+\infty]{} 0\,.$$

This yields (10). Hence $N_1(T) = 1$.

<u>Proof of "$(\tilde{T}_{i,n})$ is **WD**"</u> : We have to prove (7) and

$$\lim_{N\to+\infty}\lim_{t\to 0}\limsup_{n\to+\infty}\sup_{N\le m\le[nt]} n\left\|\tilde{T}_{0,n}\sum_{k=N}^{m}\mathbb{E}(\tilde{T}_{k,n}|\mathcal{M}_0)\right\|_1 = 0\,. \tag{13}$$

The proof of (13) is a straightforward consequence of (10).

Let us study (7). Using the dependence properties of the sequences $(X_i)_{i\in\mathbb{Z}}$ and $(\varepsilon_i)_{i\in\mathbb{Z}}$, we get

$$\mathbb{E}(S_n(t)|\mathcal{M}_0) = \sum_{k=1}^{[nt]}\mathbb{E}(\tilde{\varphi}_n(X_k)|\mathcal{M}_0)\,. \tag{14}$$

On one hand, we have

$$\|\mathbb{E}(\tilde{\varphi}_n(X_k)|\mathcal{M}_0)\|_1 \le (\|d\varphi_n\|)\phi_k = \mathcal{O}\left(\frac{\alpha(n,h_n)}{nh_n}\phi_k\right)\,. \tag{15}$$

On the other hand we have

$$\|\mathbb{E}(\tilde{\varphi}_n(X_k)|\mathcal{M}_0)\|_1 \le 2\mathbb{E}\,|\varphi_n(X_k)| = \mathcal{O}\left(\frac{\alpha(n,h_n)}{nh_n}\frac{h_n}{\beta(n,h_n)}\right)\,. \tag{16}$$

Hence, using the dominated convergence theorem, we deduce from (14), (15) and (16) that for any fixed $t>0$

$$\limsup_{n\to+\infty}\frac{1}{t}\|\mathbb{E}(S_n(t)|\mathcal{M}_0)\|_1 = 0\,,$$

and consequently

$$\lim_{t\to 0}\limsup_{n\to+\infty}\frac{1}{t}\|\mathbb{E}(S_n(t)|\mathcal{M}_0)\|_1 = 0\,.$$

Hence the sequence $(\tilde{T}_{i,n})$ is **WD**.

<u>Proof of "$(\tilde{T}_{i,n})$ is **EQ**" and $N_2(T)=1$</u> : We use here [3, Definition 3, page 9]. Hence, we are going to prove that $(\tilde{T}_{i,n})$ is 1-EQ. For this, we have to prove the two following points :

$$\limsup_{n\to+\infty} n\mathbb{E}(\tilde{T}_{0,n}^2) \le C \tag{17}$$

for some positive constant C.

$$\lim_{t\to 0}\limsup_{n\to+\infty}\frac{1}{t}\mathbb{E}\left(\sum_{k=1}^{[nt]}\tilde{T}_{k,n}^2\left(1\wedge\overline{S}_{k-1,n}\right)\right) = 0\,, \tag{18}$$

where $\overline{S}_{k-1,n} := \max\left(|S_{1,n}|,\ldots,|S_{k-1,n}|\right)$. Let us first study (17). For any real u (including infinity), define $\Gamma_n(u) := n\mathbb{E}\left(\tilde{T}_{0,n}^2\mathbb{1}_{\tilde{T}_{0,n}\le u}\right)$. We have then $n\mathbb{E}(\tilde{T}_{0,n}^2) = \Gamma_n(+\infty)$. The study of this term does not depend on the dependence property of the sequence $(X_i)_{i\in\mathbb{Z}}$. Hence, we know from Grégoire and Hamrouni [10] that $\Gamma_n(+\infty)$ converges, as n tends to infinity, to a finite limit $\Gamma(+\infty)$ which depends on a, b, and on the relative positions of z_1, z_2 and 0. It concludes the proof of (17).

It remains to prove (18). Using the stationarity and dependence properties of $(X_i)_{i\in\mathbb{Z}}$ and $(\varepsilon_i)_{i\in\mathbb{Z}}$, we have

$$\mathbb{E}\left(\tilde{T}_{k,n}^2\,(1\wedge\overline{S}_{k-1,n})\right) \;\leq\; \sum_{i=1}^{k-1}\mathbb{E}\left(\tilde{T}_{k,n}^2\,|\tilde{T}_{i,n}|\right)=\sum_{i=1}^{k-1}\mathbb{E}\left(\tilde{T}_{i,n}^2\,|\tilde{T}_{0,n}|\right)\,.$$

Let us bound $\mathbb{E}\left(\tilde{T}_{i,n}^2\,|\tilde{T}_{0,n}|\right)$. There exists some constant C such that for any i in $\mathbb{N}^*$,

$$\mathbb{E}\left(\tilde{T}_{i,n}^2\,|\tilde{T}_{0,n}|\right)\;\leq\; C\,\frac{\alpha(n,h_n)}{nh_n}\left(\mathrm{Cov}\left(\tilde{U}_{i,n},\tilde{U}_{0,n}\right)+\left(\mathbb{E}\tilde{U}_{0,n}\right)^2\right)\,. \tag{19}$$

Hence, as $\alpha(n,h_n)/(nh_n)$ tends to $L_5<+\infty$ as n tends to infinity, there exists some constant C such that

$$\frac{1}{t}\mathbb{E}\left(\sum_{k=1}^{[nt]}\tilde{T}_{k,n}^2\,(1\wedge\overline{S}_{k-1})\right)\leq\frac{C}{t}\sum_{k=1}^{[nt]}\sum_{i=1}^{k-1}\left(\mathrm{Cov}\left(\tilde{U}_{i,n},\tilde{U}_{0,n}\right)+\left(\mathbb{E}\tilde{U}_{0,n}\right)^2\right)\,. \tag{20}$$

<u>Study of $\frac{1}{t}\sum_{k=1}^{[nt]}\sum_{i=1}^{k-1}\mathrm{Cov}\left(\tilde{U}_{i,n},\tilde{U}_{0,n}\right)$.</u>

We first notice that

$$\frac{1}{t}\sum_{k=1}^{[nt]}\sum_{i=1}^{k-1}\mathrm{Cov}\left(\tilde{U}_{i,n},\tilde{U}_{0,n}\right)\leq n\sum_{i\geq 1}\mathrm{Cov}\left(\tilde{U}_{i,n},\tilde{U}_{0,n}\right)\,. \tag{21}$$

Now, using the dependence property of $(X_i)_{i\in\mathbb{Z}}$, there exists a constant C such that

$$n\,\mathrm{Cov}\left(\tilde{U}_{i,n},\tilde{U}_{0,n}\right)\leq C\,n\left(\frac{\alpha(n,h_n)}{nh_n}\right)^2\frac{h_n}{\beta(n,h_n)}\,\phi_i\,. \tag{22}$$

Assumption (1) yields that for each i, $n\,\mathrm{Cov}\left(\tilde{U}_{i,n},\tilde{U}_{0,n}\right)$ tends to 0 as n tends to infinity. Hence, as $\sum_{i\geq 1}\phi_i<+\infty$, the dominated convergence theorem yields : $\frac{1}{t}\sum_{k=1}^{[nt]}\sum_{i=1}^{k-1}\mathrm{Cov}\left(\tilde{U}_{i,n},\tilde{U}_{0,n}\right)$ tends to 0 as n tends to infinity.

<u>Study of $\frac{1}{t}\sum_{k=1}^{[nt]}\sum_{i=1}^{k-1}\left(\mathbb{E}\tilde{U}_{0,n}\right)^2$.</u>

We have $\mathbb{E}\tilde{U}_{0,n}=\mathcal{O}\left(\frac{h_n}{\beta(n,h_n)}\right)$. Hence there exists some constant C such that

$$\sum_{i=1}^{k-1}\left(\mathbb{E}\tilde{U}_{0,n}\right)^2\leq C\,k\left(\frac{h_n}{\beta(n,h_n)}\right)^2\,. \tag{23}$$

Hence for any fixed $t > 0$,

$$\frac{1}{t}\sum_{k=1}^{[nt]}\sum_{i=1}^{k-1}\left(\mathbb{E}\tilde{U}_{0,n}\right)^2 \leq \frac{1}{t}\frac{[nt]\,(1+[nt])}{2}\mathcal{O}\left(\frac{1}{n^2}\right) = t\ \ \mathcal{O}\,(1)\ .$$

Then, as $t \to 0$, we deduce (18). It concludes the proof of $(\tilde{T}_{i,n})$ is 1-EQ.

Hence we have proved that the assumptions of Corollary 5.1 are satisfied. Recall that $\Gamma_n(u) = n\,\mathbb{E}\left(\tilde{T}_{0,n}^2\mathbb{1}_{\tilde{T}_{0,n}\leq u}\right)$. We know from Grégoire and Hamrouni [10] that $\Gamma_n(u)$ tends to a limit $\Gamma(u)$ where Γ is a distribution function. In fact, using their proof, we get the following stronger result :

$$\mathbb{E}\left|n\tilde{T}_{0,n}^2\mathbb{1}_{\tilde{T}_{0,n}\leq u} - \Gamma(u)\right| \xrightarrow[n\to+\infty]{} 0\,. \tag{24}$$

Moreover,

$$\left\|\mathbb{E}\left(\frac{1}{t}\sum_{k=1}^{[nt]}\tilde{T}_{k,n}^2\mathbb{1}_{\tilde{T}_{k,n}\leq x} - \Gamma(x)|\mathcal{M}_0\right)\right\|_1$$

is bounded by

$$\sum_{k=1}^{[nt]}\left\|\frac{1}{nt}\mathbb{E}\left(n\tilde{T}_{k,n}^2\mathbb{1}_{\tilde{T}_{k,n}\leq x} - \Gamma(x)\,|\mathcal{M}_0\right)\right\|_1 + \frac{[nt]}{nt}\Gamma(x)\left(\frac{nt}{[nt]} - 1\right)\,. \tag{25}$$

Let t be fixed. The second term in (25) tends to 0 as n tends to infinity. Let us now study the first term. Using (24) and the stationarity of $(X_i)_{i\in\mathbb{Z}}$, we have that

$$\sum_{k=1}^{[nt]}\left\|\frac{1}{nt}\mathbb{E}\left(n\tilde{T}_{k,n}^2\mathbb{1}_{\tilde{T}_{k,n}\leq x} - \Gamma(x)\,|\mathcal{M}_0\right)\right\|_1 \leq \frac{[nt]}{nt}\mathbb{E}\left|n\tilde{T}_{0,n}^2\mathbb{1}_{\tilde{T}_{0,n}\leq x} - \Gamma(x)\right|\,, \tag{26}$$

which tends to zero as n tends to infinity. Hence, we get that

$$\lim_{t\to0}\limsup_{n\to+\infty}\left\|\mathbb{E}\left(\frac{1}{t}\sum_{k=1}^{[nt]}\tilde{T}_{k,n}^2\mathbb{1}_{\tilde{T}_{k,n}\leq x} - \Gamma(x)|\mathcal{M}_0\right)\right\|_1 = 0\,.$$

Therefore we get $\mathbf{S_1}$ which implies the convergence in distribution of $S_n(1)$. It concludes the proof of the 2-dimensional convergence. Now, to derive the limit process $\mathcal{M}$, we proceed as in Section 5 of Grégoire and Hamrouni [10], starting from Kolmogorov's formula (6). We get then the 2-dimensional convergence of the process $\mathcal{M}_n$ to the bilateral compound Poisson process $\mathcal{M}$ defined as

$$\mathcal{M}(z) = -\lambda_3\gamma(\tau)|z| + \lambda_1\mathcal{N}(z)$$

with $\mathcal{N}(z)$ defined by (4).

5.2 Tightness

Let us state the tightness result. Using [3, Remark 5 and Proposition 2, page 9], we get the tightness, as $(\tilde{T}_{i,n})$ is WD and 1-EQ. Now, from the 2-convergence and from the tightness, we deduce the weak convergence of the process $(\mathcal{M}_n)$.

Concluding remarks

In this paper, we studied a change point problem for dependent data. The interest of studying what happens for dependent data is obvious, as many commonly used models are not independent. In this paper, our model was the following regression model :

$$Y_i = m(X_i) + \sigma(X_i)\varepsilon_i \,,$$

where $(\varepsilon_i)_{i\in\mathbb{Z}}$ is independent and identically distributed, and independent of $(X_i)_{i\in\mathbb{Z}}$, which is assumed to be ϕ-dependent. This notion of dependence covers many interesting models. Using local linear smoothing, we clarify in this paper that the limiting process of a local dilated-rescaled version of the jump estimate process is the same as for independent data.

6 Appendix A : Proof of Theorem 4.2

We first prove the two following lemmas.

Lemma 6.1 *Assume that the sequence $(X_i)_{i\in\mathbb{Z}}$ is ϕ-dependent with $\sum_{i=1}^{+\infty} \phi(i) < +\infty$.*

(i) Under H_1 and H_2, if we define $S_l^+(x)$ by $\sum_{i=1}^{n}(x - X_i)^l K_i^+(x)$, then

$$S_l^+(\tau + h_n y) = n h_n^{l+1} f(\tau)\left(K_l^+ + o_P(1)\right).$$

(ii) Under H_1, H_2, and $H3$,

$$S_l^+(\tau + h_n y) - S_l^+(\tau) = \mathcal{O}\left(\frac{n h_n^{l+1}}{\beta(n, h_n)}\right) + \mathcal{O}_P\left(\sqrt{\frac{n h_n^{2l+1}}{\beta(n, h_n)}}\right).$$

(iii) Let $m_+(\tau) := \lim_{t\to\tau,\, t>\tau} m(t)$. Under H_1,

$$D_l^+(\tau) := \sum_{i=1}^{n}(\tau - X_i)^l K_+\left(\frac{\tau - X_i}{h_n}\right)(Y_i - m_+(\tau)) = \mathcal{O}\left(n h_n^{l+2}\right) + \mathcal{O}_P\left(\sqrt{n h_n^{2l+1}}\right).$$

Lemma 6.2 *Assume that the sequence $(X_i)_{i\in\mathbb{Z}}$ is ϕ-dependent with $\sum_{i=1}^{+\infty} \phi(i) < +\infty$.*

(i) Under H_1 and H_2,

$$\sum_{i=1}^{n} w_i^+(\tau + h_n y) = n^2 h_n^4 f^2(\tau)(1 + o_P(1)).$$

(ii) Under H_1, H_2 and $H3$,

1)

$$\frac{\sum_{i=1}^n w_i^+(\tau + h_n y) - w_i^+(\tau)}{\sum_{i=1}^n w_i^+(\tau + h_n y)} = \mathcal{O}_P\left(\frac{1}{\beta(n, h_n)} + \frac{1}{\sqrt{n h_n \beta(n, h_n)}}\right),$$

2)

$$\begin{aligned}\alpha(n, h_n)\tfrac{\sum_{i=1}^n (w_i^+(\tau + h_n y) - w_i^+(\tau))(Y_i - m_+(\tau))}{\sum_{i=1}^n w_i^+(\tau + h_n y)} &= \tfrac{1 + o_P(1)}{f(\tau)}\mathcal{M}_n^+(z) \\ &+ \mathcal{O}_P\left(\left[\tfrac{\alpha(n,h_n)}{\beta_n(n,h_n)} + \tfrac{\alpha(n,h_n)}{\sqrt{n h_n \beta(n,h_n)}}\right]\left[h_n + \tfrac{1}{\sqrt{n h_n}}\right]\right).\end{aligned}$$

(iii) Under H_1,

$$\frac{\sum_{i=1}^n (w_i^+(\tau)(Y_i - m_+(\tau))}{\sum_{i=1}^n w_i^+(\tau)} = \mathcal{O}_P\left(h_n^2 + \frac{1}{\sqrt{n h_n}}\right).$$

Let us write now the proofs of Lemmas 6.1 and 6.2.

Proof of Lemma 6.1

(i) The proof is a rather straightforward modification of the one of Theorem 2.1 in Ango Nze and Prieur [1] (see also [15]).

(ii) Thanks to Bienaymé-Tchebycheff inequality, we write:

$$S_l^+(\tau + h_n y) - S_l^+(\tau) = \mathbb{E}\left(S_l^+(\tau + h_n y) - S_l^+(\tau)\right) + \mathcal{O}_P\left(\sqrt{\text{Var}\ \left(S_l^+(\tau + h_n y) - S_l^+(\tau)\right)}\right). \quad (27)$$

For sake of simplicity, let assume that $y \geq 0$. The proof for $y < 0$ is then similar.

We have:

$$\begin{aligned}\mathbb{E}\left(S_l^+(\tau + h_n y) - S_l^+(\tau)\right) &= n h_n^{l+1} \int \left((x + y)^l K_+(x + y) - x^l K_+(x)\right) f(\tau - h_n x) dx \\ &= n h_n^{l+1} \int_{-1-y}^{-1} (x + y)^l K_+(x + y) f(\tau - h_n x) dx \\ &+ n h_n^{l+1} \int_{-1}^{-y} \left((x + y)^l K_+(x + y) - x^l K_+(x)\right) f(\tau - h_n x) dx \\ &- n h_n^{l+1} \int_{-y}^{0} x^l K_+(x) f(\tau - h_n x) dx. \end{aligned} \quad (28)$$

Under assumptions H_1 and H_2, $nh_n \xrightarrow[n\to\infty]{} \infty$ and $\frac{h_n}{\beta(n,h_n)} \xrightarrow[n\to\infty]{} 0$, then

$$\begin{aligned}
\int_{-1-y}^{-1}(x+y)^l K_+(x+y) f(\tau - h_n x)dx &= \int_0^y (x-1)^l K_+(x-1) f(\tau - h_n(x-1-y))dx \\
&= \int_0^y \left((-1)^l K_+(-1) + \mathcal{O}(x)\right)(f(\tau)+o(1))\,dx \\
&= \mathcal{O}(y^2) = \mathcal{O}\left(\frac{1}{\beta(n,h_n)^2}\right) .
\end{aligned}$$

Similarly,

$$\int_{-y}^{0} x^l K_+(x) f(\tau - h_n x)dx = \mathcal{O}\left(\frac{1}{\beta(n,h_n)}\right) ,$$

and

$$\int_{-1}^{-y}\left((x+y)^l K_+(x+y) - x^l K_+(x)\right) f(\tau - h_n x)dx = \mathcal{O}(y) = \mathcal{O}\left(\frac{1}{\beta(n,h_n)}\right).$$

Hence,

$$\mathbb{E}\left(S_l^+(\tau + h_n y) - S_l^+(\tau)\right) = \mathcal{O}\left(\frac{nh_n^{l+1}}{\beta(n,h_n)}\right).$$

We also have:

$$\begin{aligned}
\text{Var}\ \left(S_l^+(\tau+h_n y) - S_l^+(\tau)\right) &= n\text{Var}\ \left((\tau + h_n y - X_1)^l K_1^+(\tau+h_n y) - (\tau - X_1)^l K_1^+(\tau)\right) \\
&+ 2\sum_{k=2}^{n}(n-k+1)\text{Cov}\left(\Psi_k(\tau+h_n y) - \Psi_k(\tau), \Psi_1(\tau+h_n y) - \Psi_1(\tau)\right) , \quad (29)
\end{aligned}$$

where $\Psi_i(t) = (t - X_i)^l K_+\left(\frac{t - X_i}{h_n}\right)$.

Define $C_{k,1}$ by

$$\text{Cov}\left(\Psi_k(\tau+h_n y) - \Psi_k(\tau), \Psi_1(\tau+h_n y) - \Psi_1(\tau)\right).$$

Using same kind of decomposition as (28), we get

$$n\text{Var}\ \left((\tau + h_n y - X_1)^l K_1^+(\tau + h_n y) - (\tau - X_1)^l K_1^+(\tau)\right) = \mathcal{O}\left(\frac{nh_n^{2l+1}}{\beta(n,h_n)}\right).$$

Now, using the dependence properties of the sequence $(X_i)_{i\in\mathbb{Z}}$, we get the following upper bound for the covariance terms :

$$\sum_{k=2}^{n}(n-k+1)\,C_{k,1} = \mathcal{O}\left(\frac{n\,h_n^{2l+1}}{\beta(n,h_n)}\sum_{i\geq 1}\phi_i\right) .$$

Therefore, as $\sum_{i\geq 1}\phi_i < +\infty$,

$$\text{Var}\ \left(S_l^+(\tau + h_n y) - S_l^+(\tau)\right) = \mathcal{O}\left(n\,h^{2l+1}\,\beta^{-1}(n,h_n)\right) .$$

We then conclude by using equality (27).

(iii) Using once more Bienaymé-Tchebycheff's inequality, we have

$$D_l^+(\tau) = \mathbb{E}D_l^+(\tau) + \mathcal{O}_P\left(\sqrt{\mathrm{Var}\ D_l^+(\tau)}\right) . \tag{30}$$

We have

$$\begin{aligned}\mathbb{E}D_l^+(\tau) &= nh_n^{l+1}\int x^l K_+(x)(m(\tau - h_n x) - m_+(\tau))f(\tau - h_n x)dx \\ &= \mathcal{O}\left(nh_n^{l+2}\right) .\end{aligned}$$

Let $p_n^l(X_i) := (\tau - X_i)^l K_+\left(\frac{\tau - X_i}{h_n}\right)(m(X_i) - m_+(\tau))$.

Using the stationarity and the independence properties of the sequences $(\varepsilon_i)_{i\in\mathbb{Z}}$ and $(X_i)_{i\in\mathbb{Z}}$, we write

$$\begin{aligned}\mathrm{Var}\ D_l^+(\tau) = n\,\mathrm{Var}\ &\left((\tau - X_1)^l K_+\left(\frac{\tau - X_1}{h_n}\right)(Y_1 - m_+(\tau))\right) \\ &+ 2\sum_{k=2}^{n}(n-k+1)\mathrm{Cov}\left(p_n^l(X_1), p_n^l(X_k)\right) .\end{aligned} \tag{31}$$

Then

$$\begin{aligned}&n\mathrm{Var}\ \left((\tau - X_1)^l K_+\left(\frac{\tau - X_1}{h_n}\right)(Y_1 - m_+(\tau))\right) \\ &\leq nh_n^{2l+1}\int x^{2l}K_+^2(x)\left((m(\tau - h_n x) - m_+(\tau))^2 + \sigma^2(\tau - xh_n)\right)f(\tau - h_n x)dx = \mathcal{O}\left(nh_n^{2l+1}\right) .\end{aligned} \tag{32}$$

Using the dependence property of the sequence $(X_i)_{i\in\mathbb{Z}}$, we get

$$\left|2\sum_{k=2}^{n}(n-k+1)\mathrm{Cov}\left(p_n^l(X_1), p_n^l(X_k)\right)\right| \leq 2h_n^{2l+3}\sum_{k=1}^{n-1}(n-k)\phi_k = \mathcal{O}\left(n\,h_n^{2l+3}\sum_{k=1}^{+\infty}\phi_k\right) . \tag{33}$$

Recall that $\sum_{k=1}^{+\infty}\phi_k < +\infty$. Hence, using (32) and (33), we get $\mathrm{Var}\ D_l^+(\tau) = \mathcal{O}\left(nh_n^{2l+1}\right)$. Together with (30) and (31) it yields (iii). □

Proof of Lemma 6.2

(i) It is a straightforward consequence of the decomposition

$$\sum_{i=1}^{n} w_i^+(\tau + h_n y) = S_2^+(\tau + h_n y)S_0^+(\tau + h_n y) - \left(S_1^+(\tau + h_n y)\right)^2$$

and of point (i) of Lemma 6.1.

(ii) 1) Proceeding as in the paper of Grégoire and Hamrouni [10], we can write

$$\begin{aligned}\sum_{i=1}^{n}(\omega_i^+(\tau+h_ny)-\omega_i^+(\tau)) &= \left(S_2^+(\tau+h_ny)-S_2^+(\tau)\right)S_0^+(\tau)+\left(S_1^+(\tau+h_ny)-S_1^+(\tau)\right)^2 \\ &\quad+\left(S_0^+(\tau+h_ny)-S_0^+(\tau)\right)S_2^+(\tau+h_ny) \\ &\quad-2\left(S_1^+(\tau+h_ny)-S_1^+(\tau)\right)S_1^+(\tau+h_ny)\end{aligned} \quad . \tag{34}$$

Then, using point (ii) of Lemma 6.1 and point (i) of Lemma 6.2, we get (ii) 1) of Lemma 6.2.

(ii) 2) We proceed as in the paper of Grégoire and Hamrouni [10] (proof of point c of their Lemma 5.2).

(iii) The proof follows from Theorem 2.2 in Ango Nze and Prieur [1] and from the following decomposition

$$\hat{m}^{\pm}(\tau)=\frac{\sum_{i=1}^n w_i^{\pm}(\tau)Y_i}{\sum_{i=1}^n w_i^{\pm}(\tau)} .$$

□

The result of Theorem 4.2 is deduced from Lemma 6.1 and 6.2 above, as in Grégoire and Hamrouni [10].

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