RISK INDICATORS WITH SEVERAL LINES OF BUSINESS: COMPARISON, ASYMPTOTIC BEHAVIOR AND APPLICATIONS TO OPTIMAL RESERVE ALLOCATION

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Abstract. In a multi-dimensional risk model with dependent lines of business, we propose to allocate capital with respect to the minimization of some risk indicators. These indicators are sums of expected penalties due to the insolvency of a branch while the global reserve is either positive or negative. Explicit formulas in the case of two branches are obtained for several models (independent exponential, correlated Pareto). The asymptotic behavior (as the initial capital goes to infinity) is studied. For higher dimension and several periods, no explicit expression is available. Using a stochastic algorithm, we get estimations of the allocation, compare the different allocations and study the impact of dependence.

Introduction

The current change of regulation leads the insurance industry to address new questions regarding solvency. In Europe, insurance groups will soon have to comply with new rules, namely Solvency II. In this new framework, solvency requirements may either be computed through a standard formula, or with internal models that companies are encouraged to develop.

While a bottom-up approach is used in the standard formula to aggregate risks (one first studies each small risk separately and then aggregates them with a matrix of correlation parameters), a top-down approach may be used in some internal or partial internal models to allocate economic capital: once the main risk drivers for the overall company have been identified and the global solvency capital requirement has been computed, it is necessary to split this overall buffer capital into marginal solvency capitals for each line of business, in order to penalize as equitably as possible lines of business and customers according to the share of the overall risk they represent after diversification.

Once this is done, one may also want to allocate some free additional surplus in some different regulation zones in order to avoid as far as possible that some lines of business become insolvent too often. Capital fungibility between lines of business or between entities of a large insurance group that lie in different countries is indeed limited by different entity-specific or country-specific solvency constraints. For example, when AIG experienced problems in the USA during the credit crunch, it was impossible for the group to transfer some funds from some European branches to the ones in distress. Some European branches were able to continue business almost as usual and were not impacted, while some other lines would have to get up to 37.8 billion US dollars of fresh liquidity. Sometimes, fungibility is limited due to particular profit-sharing mechanisms; it may also happen that profits are difficult to extract from lines of business of certain countries (like Argentina).

One possible way to define optimality of the global reserve allocation is to minimize the expected sum of penalties that each line of business would have to pay due to its temporary potential insolvency, following ideas presented in [11]. If one neglects discounting factors, a first approximation of this penalty is given by the time-integrated expected negative part of the surplus process. Closed-form formulas were available in the classical risk model for exponentially distributed claim amounts, which led to a semi-explicit optimal reserve allocation.

However, this approach does not take into account the dependence structure. Also, one could object that if the company is ruined at the group level (*i.e.* the sum of the surpluses is negative), then the allocation does not change anything. Following [4], we define and study risk indicators that are sums (over p time periods and d lines of business or branches) of the absolute expected loss of the k^{th} line over the j^{th} period ($k = 1, \dots, d$, $j = 1, \dots, p$) while globally the firm is safe (indicators I_1 and I_2) or while globally the firm is unsafe (indicator J) - see Section 1 for formal definitions). Once the indicators have been defined, we look for the optimal allocation $(u^1, \dots, u^d)$ - where u^k stands for the part of capital allocated to the k^{th} branch - that minimizes one of the indicators, under the constraint that $u^1 + \cdots + u^d = u$, where u is the total amount to be allocated. The indicators I_1 and I_2 take into account the loss of branches only when globally the firm is safe, while J takes into account the loss of branches when globally the firm is unsafe. As we shall see, this will lead to very different allocations, they are designed for different purposes.

Here we consider very basic insurance discrete time models for different lines, and we try to study the optimal reserve allocation. We shall see that considering the indicator I_1 or I_2 is a completely different approach from the economic capital allocation problem, where one tries to penalize contributions to the overall risk and to diversification effect.

As noticed in [6], *different capital allocations must in some sense correspond to different questions that can be asked within the context of risk management.* A review on various ways to allocate capital may be found in [5]. Also J.-P. Laurent [10] gives a global view of capital allocation and exhibits connections between several theories (including Euler, Shapley-Aumann and Pareto optimal allocations). Let us remark that even if our purpose implies a *risk driven* allocation of capital, we do not refer explicitly to risk measure theory (see [3, 8, 9]). An example is the TVaR allocation principle (which appears also as a particular case of [6], see equation (54)), according to which each line of business k has to pay for cost of capital corresponding to average marginal loss for line k given that things go wrong for the global company (the VaR of the sum is overshot). This allocation principle is similar in spirit to the minimization of the indicator J, while the indicators I_1 and I_2 are placed in a different context and relevant when the global company is safe. With indicators I_1 or I_2, we are more interested in the stability and liquidity issues with limited capital fungibility. When the global company is ruined, this allocation has no importance (even if in practice, one may break the mutual solidarity principle and decide to run separately healthy lines). Finally, note that we do not take into account the penalties in the overall reserve process; in theory the line of business or the global company would have to pay the penalty upfront, and this could accelerate ruin.

The present article is an attempt to explore the quantitative behavior of the allocation procedure, based on the minimization of a risk indicator. While Dhaene et al. (see Theorem 3 in [6]) show that the optimization problem leads to compute quantile functions and comonotonic sums related to each risk and to the specific form of the indicator, we propose explicit computations in dimension 2 and provide a simulation study in higher dimension and for multi-periodic models. The simulation study is made possible through the algorithmic procedure proposed in [4]. Let us emphasize that in [6], the question of the effective computation of the solution to the optimization problem is not addressed.

Our main theoretical results are Theorem 2.7 and Theorem 2.5 from which the asymptotic (as u goes to infinity) behavior of the allocation may be derived for large classes of independent distributions in dimension 2. In the present paper, the asymptotic behavior is described for Generalized Pareto Distributions. These asymptotic results may serve as benchmark. Then, we provide a simulation study in higher dimension and over several periods. This study shows that the method is numerically tractable and confirms that the choice of the right indicator is crucial according to the nature of the capital to be allocated as well as the aims of the allocation.

Our paper is organized as follows. In Section 1, we present the main notations and give the formal definitions of the risk indicators. In Sections 2 and 3, we tackle the case with two lines of business ($d = 2$) and one period ($p = 1$). We derive semi-explicit formulas and/or asymptotic results in the cases where aggregate claim amounts are independent exponentials, independent Generalized Pareto Distributions (GPD), correlated GPD, for the minimization of the indicators. Note that throughout the article, we do not consider time-dependencies that would be bred by claim development issues: we neglect reserving aspects and assume that claim settlement is immediate. Section 4 consists in a simulation study for the case treated in Section 2 and Section 3, using the algorithm developed in [4]. This section helps for benchmarking the parameters of the algorithm. In Section 5, models in higher dimension ($d = 10$), still for one period, are considered. In Section 6, we investigate time horizons $p > 1$ and $d = 2, \dots, 5$ lines of business. For the models considered in these two last sections, semi-explicit formulas are generally not available. This is the subject of a further work to develop theoretical results for $d > 2$ and $p > 1$.

1. Main definitions and notation

Throughout the paper, X_j^k denotes the (non negative) claim amount of the k^{th} business line during the j^{th} period, Y_j^k is the aggregate claim amount of the k^{th} business line over the first j periods:

$$Y_j^k := \sum_{i=1}^{j} X_i^k,$$

and R_j^k is the reserve of line k at time j:

$$R_j^k := u^k + j \times c^k - Y_j^k = u^k + \sum_{i=1}^{j} (c^k - X_j^k)$$

where u^k is the initial reserve of line k and c^k the premium paid to the k^{th} line at each time period.

Let u denote the total amount to be allocated between the d branches. We look for the optimal allocation $(u^1, \dots, u^d)$, under the constraint $u^1 + \dots + u^d = u$, minimizing

$$I_1 := \sum_{j=1}^{p\wedge\tau} \sum_{k=1}^{d} \mathbb{E}\left(|R_j^k| 1\!\!1_{\left\{R_j^k<0\right\}} 1\!\!1_{\left\{R_j^1+\cdots+R_j^d\geq 0\right\}} \right)$$

where

$$\tau := \inf\left\{ j \in \mathbb{N}^*,\ R_j^1 + \cdots + R_j^d < 0 \right\}.$$

The I_1 indicator is the *stopped* version of the indicator considered in [4]:

$$I_2 := \sum_{j=1}^{p} \sum_{k=1}^{d} \mathbb{E}\left(|R_j^k| 1\!\!1_{\{R_j^k<0\}} 1\!\!1_{\{R_j^1+\cdots+R_j^d\geq 0\}} \right).$$

Of course, in the case $p = 1$, $I_1 = I_2$. Below, I_1 is referred to as the *stopped orange area* and I_2 is referred to as the *orange area*. In the present paper, we also consider the alternative risk indicator, called expected *violet area*:

$$J := \sum_{j=1}^{p} \sum_{k=1}^{d} \mathbb{E}\left(|R_j^k| 1\!\!1_{\left\{R_j^k<0\right\}} 1\!\!1_{\left\{R_j^1+\cdots+R_j^d\leq 0\right\}} \right).$$

The three studied cases are represented in Figure 1 in the particular case when $d = 2$. At each unit time j, the violet (resp. the blue) dot-line stands for the reserve of the first (resp. the second) branch while the bold plain line represents the total reserve $R_j^1 + R_j^2$. When the firm is unsafe, the total reserve is negative and the loss is taken into account in the *violet area*. When the firm is globally safe but with one of the two branches with a negative reserve -in dot line-, the loss is taken into account in the *orange area*.

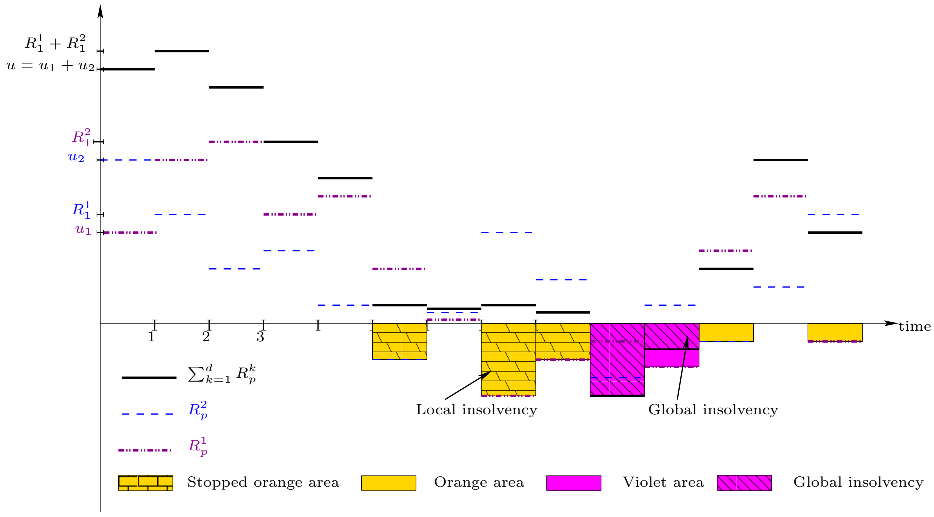

FIGURE 1. Representation of the three indicators in the particular case $d = 2$. The *violet area* corresponds to J, the *orange area* to I_2 and the shaded orange area to I_1.

In the case when $p = 1$, these three indicators may be considered as particular cases of *Dhaene et al.*'s [6] general framework which resumes as the minimization of

$$\sum_{j=1}^{d} \mathbb{E}\left[\xi_j D\left(\frac{X_j - u^j}{v_j}\right)\right]$$

where X_j stands for the loss of the j^{th} branch, (v_j) for a weight sequence and D is a deviation function.

Our proposal corresponds to take $\xi_j = \mathbb{1}_{\{S \le u\}}$ or $\xi_j = \mathbb{1}_{\{S \ge u\}}$ (where S is the aggregate loss) and $D(x) = x^+$ (see Equations (31), (69), (72) and (74) in [6]). As mentioned in the introduction, the effective computation of the solution to the minimization problem is not addressed in [6].

2. Time horizon $p = 1$, $d = 2$ lines

In this section, we consider the one-period problem with two lines of business. So that, we will drop the j index:

$$R^k = u^k + c^k - X^k,$$

where for line $k = 1, 2$, u^k is the initial reserve, c^k is the premium and X^k is the aggregate claim amount during the first period. We start with a basic example with a discrete distribution. It illustrates the fact that the use of the *orange area* may lead to inconsistent allocation, for example, if the total capital u is too small. Then we consider more meaningful models for the claims, for which we may derive explicit solutions.

2.1. A basic example.

The following example is very basic but a caricature. It was chosen to highlight the fact that the expected *orange area* must not be used to penalize lines of business according to their risk contributions. Imagine that $c^1 = c^2 = 10$, that $X^1 = 1000Y^1$ and $X^2 = 20Y^2$, where

Y^1 and Y^2 are independent Bernoulli random variables with parameter 1/2, and that $u = 10$. The expected *orange area* is equal to $\frac{10-u^2}{4}$ for $0 \le u^2 \le u$, because the aggregate reserves are negative if $X^1 = 1000$. Consequently, the optimal reserve allocation would be the allocation of all the capital to the less risky line of business (line 2). This decision corresponds to minimizing penalties in case of temporary insolvency of one line, but is in complete disagreement with any reasonable economic capital allocation principle. Let us remark that in this example, the *violet area* equals $\frac{1}{4}(1980 - u^1)$, so that it is minimized by allocating all the capital to the most risky branch, as expected.

We remark that in this example, u and the premiums c^1 and c^2 are not consistent with the claims X^1 and X^2. Below[a], we have taken $u = 455$, $c^1 = 550$ and $c^2 = 11$. We have considered two cases: firstly Y^1 and Y^2 are independent Bernoulli variables with parameter $\frac{1}{2}$. Secondly, they are still independent Bernoulli variables but Y^1 has parameter $\frac{1}{2}$ and Y^2 has parameter $\frac{1}{3}$.

2.1.1. *$Y^1 \sim \mathcal{B}(\frac{1}{2})$, $Y^2 \sim \mathcal{B}(\frac{1}{2})$.* A direct computation gives:

$$I_1 = I_2 = J = \frac{1}{4}\left((450 - u^1)1\!\!1_{\{u^1<450\}} + (9 - u^2)1\!\!1_{\{u^2<9\}}\right).$$

The *violet* and the *orange area* are the same. Under the constraint that $u^1 + u^2 = u$, it is minimized for $5 \le u^2 \le 9$ (in this case, there is not a unique minimum).

2.1.2. *$Y^1 \sim \mathcal{B}(\frac{1}{2})$, $Y^2 \sim \mathcal{B}(\frac{1}{3})$.* A direct computation gives:

$$\begin{aligned} I_1 = I_2 &= \frac{1}{6}\left(2 * (450 - u^1)1\!\!1_{\{u^1<450\}} + (9 - u^2)1\!\!1_{\{u^2<9\}}\right), \\ J &= \frac{1}{6}\left((450 - u^1)1\!\!1_{\{u^1<450\}\}} + (9 - u^2)1\!\!1_{\{u^2<9\}}\right). \end{aligned}$$

There is a unique minimum for $I_1 = I_2$, which is reached for $u^2 = 5$ and $u^1 = 450$.

In the case of the *violet area* J, the minimum is reached for $u^2 \in [5; 9]$, the minimum is not unique.

In these two examples, the allocation proposal is in accordance with the principle that the riskier the branch, the more capital is needed. The fact that the minimum may not be unique is due to the fact that the considered distributions are discrete (see [4] for a discussion on the uniqueness of the minimum).

Let us now assume that the insurance company has previously correctly penalized lines of business according to their risks, and then wants to allocate some additional safety capital u in order to avoid temporary insolvencies of some lines. The *orange area* has been designed to answer this question.

[a]The safety loading has been set to 10% of the expected value of the claims, that is if $\mathbb{E}(X^1) = 100$ then $c^1 = 110$.

2.2. **General remarks.** We now consider more general cases, still for the period $p = 1$ and focus on the *orange area*. For the sake of simplicity, in the remainder of this section, we will consider u^k instead of $u^k + c^k$. The premium is thus included in the allocation here. Remark that for $p > 1$ this would be a non sense to include the premium in the allocation.
An easy calculation with Lagrange's multipliers shows that a non degenerate minimum (*i.e.* each line receives a positive initial reserve u^k in the optimal strategy) of I_2, under the constraint $u^1 + u^2 = u$, satisfies:

$$\mathbb{P}\left(R_1^1 < 0, R_1^1 + R_1^2 \geq 0\right) = \mathbb{P}\left(R_1^2 < 0, R_1^1 + R_1^2 \geq 0\right),$$

which can be rewritten as

$$\mathbb{P}\left(X^1 > u^1, X^1 + X^2 \leq u\right) = \mathbb{P}\left(X^2 > u^2, X^1 + X^2 \leq u\right). \tag{2.1}$$

Equation (2.1) has a solution as far as (X^1, X^2) has a continuous distribution function. Indeed in that case, the left hand side of Equation (2.1) is decreasing in u_1, equal to zero at $u_1 = u$ and to $\mathbb{P}\left(X^1 + X^2 \leq u\right)$ at $u_1 = 0$, whereas the right hand side is increasing in u_1, equal to $\mathbb{P}\left(X^1 + X^2 \leq u\right)$ at $u_1 = u$ and to zero at $u_1 = 0$.

We shall consider several probabilistic models for the claims X^1 and X^2: independent exponentials (Section 2.3), independent GPD (Section 2.5), as well as conditionally independent exponentials (Section 2.6) which are correlated Pareto distributed (GPD).

2.3. **Independent exponentials.** Assume X^1 and X^2 are independent exponential random variables with respective parameters μ^1 and μ^2. In the particular case where $\mu^1 = \mu^2$, the optimal allocation is $u^1 = u^2 = u/2$. From now on, assume that $\mu^2 > \mu^1$.

2.3.1. *Optimal allocation.*
Let us denote α and β the real numbers such that $\mu^2 = \alpha\mu^1$ and $u^1 = \beta u$, $\alpha > 1$ and $0 \leq \beta \leq 1$. Equation (2.1) leads to the optimal allocation

$$(\alpha - 1)\left(h(\beta) - h(\alpha(1-\beta))\right) - \alpha h(1) + (1+\alpha) h(\beta + \alpha - \alpha\beta) = h(\alpha) \tag{2.2}$$

where h is the function defined by $h(x) = \exp\left(-u\mu^1 x\right)$.

Proposition 2.1. *There exists a unique $\beta = \psi(\alpha, u, \mu^1) \in [0,1]$ satisfying Equation (2.2). Moreover, $0 < \psi(\alpha, u, \mu^1) \leq \frac{\alpha}{\alpha+1}$ and $\alpha \mapsto \psi(\alpha, u, \mu^1)$ is non decreasing.*

Proof. The proof is a straightforward application of the convexity of h and the implicit function theorem. □

Remark 2.2. *The fact that $\psi(\alpha, u, \mu^1)$ (and thus u^1) increases with α is consistent with the fact that if α increases then the first branch becomes riskier.*

2.3.2. *Asymptotic as the capital u goes to infinity.* Studying the asymptotic as $u \to \infty$ is useful in order to make comparison between models easier.

Proposition 2.3. *For $\alpha > 1$,*

$$\lim_{u \to \infty} \psi(\alpha, u, \mu^1) = \frac{\alpha}{\alpha + 1}.$$

Proof. Multiplying (2.2) by $h(-\beta)$, we get

$$(\alpha-1)-\alpha h(1-\beta)+(1+\alpha)h\left(\alpha(1-\beta)\right)-(\alpha-1)h\left(\alpha(1-\beta)-\beta\right)-h(\alpha-\beta)=0.$$

Remark that $\lim_{u\to\infty}\psi(\alpha,u,\mu^1)\neq 1$ since the left hand side of the previous equation has to go to 0 when u tends to ∞. Then, one has

$$\lim_{u\to\infty} -\alpha h\left(1-\psi(\alpha,u,\mu^1)\right)+(1+\alpha)h\left(\alpha\left(1-\psi(\alpha,u,\mu^1)\right)\right)-h\left(\alpha-\psi(\alpha,u,\mu^1)\right)=0,$$

and consequently $h\left(\alpha\left(1-\psi(\alpha,u,\mu^1)\right)-\psi(\alpha,u,\mu^1)\right)\underset{u\to\infty}{\longrightarrow}1$, which leads to

$$\lim_{u\to\infty}\alpha\left(1-\psi(\alpha,u,\mu^1)\right)-\psi(\alpha,u,\mu^1)=0$$

and Proposition 2.3 follows. □

Before considering the case of independent Pareto distributions, we give some asymptotic results for subexponential distributions.

2.4. **Some general results for independent and subexponential distributions.** For results on subexponential distributions, we refer to [1]. Recall that a distribution is subexponential if it is concentrated on $[0,\infty[$ and its distribution function F satisfies:

$$\frac{\overline{F^{*2}}(x)}{\overline{F}(x)}\underset{x\to\infty}{\longrightarrow}2, \tag{2.3}$$

where F^{*2} is the convolution square, that is, the distribution function of the sum of two independent variables with distribution function F. The class of subexponential distributions is denoted by $\mathcal{S}$. In [1] (Chapter IX, Proposition 1.4), it is proven that regularly varying distributions are subexponential and the following result.

Proposition 2.4 ([1])**.** *Let F be a subsexponential distribution function. Then for any $y_0\in\mathbb{R}^+$, uniformly in $y\in]0,y_0]$,*

$$\frac{\overline{F}(x-y)}{\overline{F}(x)}\underset{x\to\infty}{\longrightarrow}1.$$

We derive the following result from the properties of the subexponential class.

Theorem 2.5. *Let X and Y be two independent random variables, concentrated on $\mathbb{R}^+$. Let F_X be the distribution function of X, $u,v\in\mathbb{R}+$, $v\leq u$, $v=v(u)$. Assume that*

(1) there exists $0<\kappa_1<\kappa_2<1$ such that for u large enough, $\kappa_1\leq\frac{v}{u}\leq\kappa_2$,
(2) $F_X\in\mathcal{S}$,
(3) if[b] $v=\Theta(u)$ then

$$\frac{\overline{F}_X(v)}{\overline{F}_X(u)}\overset{u\to\infty}{=}O(1).$$

[b] Recall that for non negative functions, f and g, $f(x)$ is said to be $\Theta(g(x))$ as $x\to\infty$ if the ratio $\frac{f(x)}{g(x)}$ is bounded (away from zero and from above) as $x\to\infty$, in other words, there exist $0<K_1\leq K_2<\infty$, such that for x large enough, $K_1\leq\frac{f(x)}{g(x)}\leq K_2$.

Then we have

$$\lim_{u\to\infty} \frac{\mathbb{P}(X \geq v,\ X+Y \geq u)}{\overline{F}_X(u)} = 1. \tag{2.4}$$

Proof. We closely follow ideas from [1], Chapter IX. The fact that

$$\liminf_{u\to\infty} \frac{\mathbb{P}(X \geq v,\ X+Y \geq u)}{\overline{F}_X(u)} \geq 1$$

follows from

$$\mathbb{P}(X \geq v,\ X+Y \geq u) \geq \mathbb{P}(X \geq v,\ \max(X,Y) \geq u)$$

and

$$\begin{aligned}\mathbb{P}(X \geq v,\ \max(X,Y) \geq u) &= \mathbb{P}(X \geq u) + \mathbb{P}(X \geq v,\ Y \geq u) - \mathbb{P}(X \geq u,\ Y \geq u)\\ &= \overline{F}_X(u) + \overline{F}_X(v)\overline{F}_Y(u) - \overline{F}_X(u)\overline{F}_Y(u).\end{aligned}$$

Now, we shall prove that

$$\limsup_{u\to\infty} \frac{\mathbb{P}(X \geq v,\ X+Y \geq u)}{\overline{F}_X(u)} \leq 1.$$

Let $w \in \mathbb{R}^+$. It comes

$$\mathbb{P}(X \geq v,\ X+Y \geq u) = \mathbb{P}(X \geq v, Y \leq w,\ X+Y \geq u) + \mathbb{P}(X \geq v, Y \geq w,\ X+Y \geq u).$$

For u large enough, one gets

$$\begin{aligned}\mathbb{P}(X \geq v, Y \leq w,\ X+Y \geq u) &= \int_0^w F_Y(dy)\overline{F}_X(\max(v, u-y))\\ &= \int_0^w F_Y(dy)\overline{F}_X(u-y).\end{aligned}$$

Now, Proposition 2.4 gives

$$\lim_{u\to\infty} \frac{\mathbb{P}(X \geq v, Y \leq w,\ X+Y \geq u)}{\overline{F}_X(u)} = F_Y(w).$$

On the other hand, one has

$$\begin{aligned}\mathbb{P}(X \geq v, Y \geq w,\ X+Y \geq u) &= \int_w^{u-v} F_Y(dy)\overline{F}_X(u-y)\\ &\quad + \int_{u-v}^{\infty} F_Y(dy)\overline{F}_X(v)\\ &\leq \overline{F}_X(u-w)\overline{F}_Y(w) + \overline{F}_X(v)\overline{F}_Y(u-v),\end{aligned}$$

so that

$$\limsup_{u\to\infty} \frac{\mathbb{P}(X \geq v, Y \geq w,\ X+Y \geq u)}{\overline{F}_X(u)} \leq \overline{F}_Y(w).$$

□

2.5. **Independent Pareto.** We consider two independent Generalized Pareto Distributions (GPD) denoted by X^1 and X^2 with respective parameters $\left(\frac{1}{a}, \frac{b}{a}\right)$ and $\left(\frac{1}{a}, \frac{b}{\alpha a}\right)$, $a, b > 0$, $\alpha \geq 1$. The density and survival functions are given by: $\forall x \geq 0$

$$f_{X^1}(x) = \frac{a}{b}\left(1+\frac{x}{b}\right)^{-a-1}, \quad f_{X^2}(x) = \frac{\alpha a}{b}\left(1+\frac{\alpha x}{b}\right)^{-a-1}$$

and

$$\overline{F}_{X^1}(x) = \left(1+\frac{x}{b}\right)^{-a}, \quad \overline{F}_{X^2}(x) = \left(1+\frac{\alpha x}{b}\right)^{-a}.$$

Equation (2.1) yields in this case

$$\begin{aligned} \overline{F}_{X^1}(u^1) &- \frac{a}{b}\int_{u^1}^{u}\left(1+\frac{x}{b}\right)^{-a-1}\left(1+\frac{\alpha(u-x)}{b}\right)^{-a} dx \\ = \overline{F}_{X^2}(u^2) &- \frac{a}{b}\int_{u^2}^{u}\left(1+\frac{\alpha x}{b}\right)^{-a-1}\left(1+\frac{(u-x)}{b}\right)^{-a} dx. \end{aligned} \tag{2.5}$$

We do not know how to derive an explicit optimal allocation from this equation. Nevertheless, an asymptotic as $u \to \infty$ is reachable. We begin with a general result for independent variables.

Proposition 2.6. *Let X^1 and X^2 be two independent, continuous and non negative random variables such that (2.6) and (2.7) below are satisfied.*

$$\overline{F}_{X^1}(x) = \Theta(\overline{F}_{X^2}(x)) \text{ as } x \to \infty. \tag{2.6}$$

$$\overline{F}_{X^1}(x) = o(\overline{F}_{X^1}(t)) \text{ if } t \overset{x\to\infty}{=} o(x). \tag{2.7}$$

The solution (u^1, u^2) to Equation (2.1), under the constraint that $u^1+u^2 = u$ is such that it exists $0 < \kappa_1 < \kappa_2 < 1$ such that for all u,

$$\kappa_1 < \frac{u^1}{u} \leq \kappa_2.$$

Proof. Let $\beta(u) = \frac{u^1}{u}$. If the result were not satisfied then taking if necessary a sequence $u_k \to \infty$, we could assume that either $\beta \to 0$ or $\beta \to 1$. If $\beta \to 0$ then $\frac{u^2}{u} \to 1$ and if $\beta \to 1$, $\frac{u^2}{u} \to 0$, the problem being symetric, it suffices to prove that $\beta \not\to 1$. Let us assume that $\beta \to 1$, then $u^2 = o(u^1)$. Remark that

$$\mathbb{P}(X^1 > u^1,\ X^1 + X^2 \leq u) = \overline{F}_{X^1}(u^1) - \mathbb{P}(X^1 > u^1,\ X^1 + X^2 > u),$$

so that (2.1) rewrites:

$$\begin{aligned} &\frac{\overline{F}_{X^1}(u^1)}{\overline{F}_{X^2}(u^2)} - \frac{\mathbb{P}(X^1 > u^1,\ X^1 + X^2 > u)}{\overline{F}_{X^2}(u^2)} \\ &\quad = 1 - \frac{\mathbb{P}(X^2 > u^2,\ X^1 + X^2 > u)}{\overline{F}_{X^2}(u^2)}. \end{aligned} \tag{2.8}$$

Now,

$$\begin{aligned}
\mathbb{P}(X^1 > u^1,\ X^1+X^2 > u) &= \mathbb{P}(X^1 > u^1,\ X^2 > \sqrt{u^1},\ X^1+X^2 > u) \\
&\quad + \mathbb{P}(X^1 > u^1,\ X^2 \le \sqrt{u^1},\ X^1+X^2 > u) \\
&\le \overline{F}_{X^1}(u^1)\overline{F}_{X^2}(\sqrt{u^1}) + \overline{F}_{X^1}(u-\sqrt{u^1}) \\
&= o(\overline{F}_{X^1}(u^2)) \text{ by using (2.7)} \\
&= o(\overline{F}_{X^2}(u^2)) \text{ by using (2.6).}
\end{aligned}$$

In the same way,

$$\begin{aligned}
\mathbb{P}(X^2 > u^2,\ X^1+X^2 > u) &= \mathbb{P}(X^2 > u^2,\ X^1 > \sqrt{u},\ X^1+X^2 > u) + \\
&\quad \mathbb{P}(X^2 > u^2,\ X^1 \le \sqrt{u},\ X^1+X^2 > u) \\
&\le o(\overline{F}_{X^2}(u^2)) + \mathbb{P}(X^2 > u-\sqrt{u}) \\
&= o(\overline{F}_{X^2}(u^2)).
\end{aligned}$$

Moreover, (2.6) and (2.7) imply that $\overline{F}_{X^1}(u^1) = o(\overline{F}_{X^2}(u^2))$, so that (2.8) cannot be satisfied. □

If in addition to the hypothesis of Proposition 2.6, we work with subexponential distributions, we get asymptotic results for the ratio $\frac{u^1}{u}$ as $u \to \infty$.

Theorem 2.7. *Let X^1 and X^2 be two independent, continuous and non negative random variables such that (2.6), (2.7) and the hypothesis of Theorem 2.5 are satisfied. Then, the solution (u^1, u^2) to Equation (2.1), under the constraint that $u^1 + u^2 = u$ satisfies, as $u \to \infty$,*

$$\overline{F}_{X^1}(u^1) - \overline{F}_{X^1}(u) = \overline{F}_{X^2}(u^2) - \overline{F}_{X^2}(u). \tag{2.9}$$

Proof. The proof is deduced from Proposition 2.6 and Theorem 2.5 and uses the decomposition of $\mathbb{P}(X^1 > u^1,\ X^1+X^2 > u)$ like in the proof of Proposition 2.6. □

Proposition 2.8. *The unique solution to (2.5) satisfies*

$$\frac{\beta}{(1-\beta)} = \frac{\alpha}{[1+\alpha^a(1-\beta)^a - (1-\beta)^a]^{\frac{1}{a}}},$$

where $\beta = \lim_{u\to\infty} \frac{u^1}{u}$.

Proof. Since GPD distributions are subexponential and satisfy (2.6), (2.7) and the hypothesis of Theorem 2.5, the proof is a straightforward consequence of Theorem 2.7. □

2.6. **Correlated Pareto.** In this section, we consider a model inspired from [2]. Let X be a random variable with exponential distribution $\mathcal{E}(\Theta)$, where the parameter Θ is $\Gamma(a,b)$ distributed. Recall that the resulting mixed survival function of X is given by

$$1 - F_X(x) = \int_0^\infty e^{-\theta x} f_\Theta(\theta) d\theta = \left(1 + \frac{x}{b}\right)^{-a}, \quad x \ge 0.$$

In other words, X has a Generalized Pareto distribution (GPD).

2.6.1. *Explicit solution for fixed u.* Let X^1 and X^2 be two GPD according to the above model. Conditionally to μ^1, we assume that $X^1 \sim \mathcal{E}(\mu^1)$, $X^2 \sim \mathcal{E}(\alpha\,\mu^1)$ with $1 < \alpha$, X^1 and X^2 are independent. We assume moreover that $\mu^1 \sim \Gamma(a, b)$, $a, b > 0$. Following [2], we have that the dependence structure of the X^i's is given by a survival Clayton copula.
Conditionally to μ^1 equation (2.1) leads to:

$$(\alpha-1)h(\beta)-\alpha h(1)+h(\alpha-\alpha\beta+\beta) = (\alpha-1)h(\alpha(1-\beta))-\alpha h(\alpha-\alpha\beta+\beta)+h(\alpha).$$

We now integrate with respect to μ^1 and get the same equation replacing the function h by $s(x) = (1 + x\frac{u}{b})^{-a}$. Hence this equation can be rewritten as $f(\alpha, \beta, u) = 0$ with

$$\begin{aligned} f(\alpha,\beta,u) &= (\alpha-1)\left(s(\beta) - s\left(\alpha(1-\beta)\right)\right) \\ &\quad -\alpha s(1) + (\alpha+1)s\left(\alpha-\alpha\beta+\beta\right) - s(\alpha). \end{aligned} \tag{2.10}$$

Proposition 2.9. *There exists a unique $\beta = \Phi(\alpha, u, a, b) \in [0, 1]$ satisfying equation (2.10). Moreover, $0 < \Phi(\alpha, u, a, b) \le \frac{\alpha}{\alpha+1}$.*

Proof. We remark that $f(\alpha, 0, u) = -f(\alpha, 1, u) = (\alpha-1)+s(\alpha)-\alpha s(1) \ge 0$ as by convexity of s, one has

$$\frac{s(\alpha)-s(1)}{\alpha-1} \ge s(1) - s(0)\,.$$

Moreover one has

$$\begin{aligned} \frac{b}{a(\alpha-1)u}\frac{\partial f}{\partial\beta} &= -g(\beta) + (\alpha+1)g(\alpha-\alpha\beta+\beta) - \alpha g(\alpha(1-\beta)) \\ &= -g(\beta) + g(\alpha-\alpha\beta+\beta) + \alpha g(\alpha-\alpha\beta+\beta) - \alpha g(\alpha(1-\beta)) \\ &\le 0 \end{aligned} \tag{2.11}$$

with $g(x) = (1 + x\frac{u}{b})^{-(a+1)}$, as g is decreasing. Hence there exists a unique $\beta_0 \in [0, 1]$ such that $f(\alpha, \beta_0, u) = 0$. We also have

$$f(\alpha, \frac{\alpha}{\alpha+1}, u) = s(\gamma) - s(\alpha) + \alpha s(\gamma) - \alpha s(1) \tag{2.12}$$

with $\gamma = \frac{2\alpha}{\alpha+1}$. We remark that $1 < \gamma < \alpha$, hence using the convexity of s we have

$$s(\gamma) - s(\alpha) \le (s(1) - s(\gamma))\frac{\alpha-\gamma}{\gamma-1}$$

which implies that $f(\alpha, \frac{\alpha}{\alpha+1}, u) \le 0$ hence $\beta_0 \le \frac{\alpha}{\alpha+1}$. □

2.6.2. *Asymptotic as the capital u goes to infinity.* One gets the following result, considering the asymptotic as u goes to ∞ :

Proposition 2.10. *For $\alpha > 1$, $\lim_{u\to\infty} \Phi(\alpha, u, \mu^1) = \Phi_0$ is the solution to:*

$$(\alpha-1)\left(\Phi_0^{-a} - (\alpha(1-\Phi_0))^{-a}\right) - \alpha + (\alpha+1)(\alpha-\alpha\Phi_0+\Phi_0)^{-a} - \alpha^{-a} = 0. \tag{2.13}$$

We have $\Phi_0 < \frac{\alpha}{\alpha+1}$.

Proof. Equation (2.10) is equivalent, when $u \to \infty$ to

$$(\alpha-1)\left(\beta^{-a} - (\alpha(1-\beta))^{-a}\right) - \alpha + (\alpha+1)\left(\alpha-\alpha\beta+\beta\right)^{-a} - \alpha^{-a} = 0. \tag{2.14}$$

One wants to prove that the solution of (2.14) above satisfies $\beta < \frac{\alpha}{\alpha+1}$ when $\alpha > 1$. Indeed, define

$$g(\beta) = (\alpha-1)\beta^{-a} - \alpha + (\alpha+1)(\alpha - \alpha\beta + \beta)^{-a} - (\alpha-1)\alpha^{-a}(1-\beta)^{-a} - \alpha^{-a}.$$

One has $g(0^+) = +\infty$ and $g(1^-) = -\infty$. Straightforward computations prove that g is decreasing in β and that $g(\frac{\alpha}{\alpha+1}) < 0$. This proves that the solution Φ_0 of (2.14) satisfies $\Phi_0 < \frac{\alpha}{\alpha+1}$. □

3. Comparison with an indicator designed for crisis

Recall that the *violet area* is given by the indicator

$$J(u^1, \ldots, u^d) = \sum_{j=1}^{p} \sum_{k=1}^{d} E\left(|R_j^k| \mathbb{1}_{\left\{R_j^k < 0\right\}} \mathbb{1}_{\left\{R_j^1 + \cdots + R_j^d \leq 0\right\}} \right).$$

As mentioned in the introduction, minimizing the indicator J under the constraint $u^1 + \cdots + u^d = u$ corresponds to a more classical economic capital allocation principle, closely related, in spirit, to the TVaR based allocation. In this section, we focus on this allocation based on the *violet area.*

As the computations are similar to the ones corresponding to the *orange area*, we do not go into the details. We begin with the *independent exponential distributions* case and then consider the *conditionally exponential independent distributions* case. Asymptotic results for the *independent GPD* case are also established. For this last purpose, we study the asymptotic behavior of probabilities for subexponential distributions. A non degenerate minimum for J, under the constraint $u^1 + u^2 = u$ is reached for u^1 and u^2 satisfying:

$$\mathbb{P}(X^1 \geq u^1,\ X^1 + X^2 \geq u) = \mathbb{P}(X^2 \geq u^2,\ X^1 + X^2 \geq u). \tag{3.1}$$

We have the following implicit solutions to the optimal allocation issue (recall that $u^1 = \beta u$):

- **Independent exponential distributions:**

$$(\alpha+1)e^{-u\mu^1(\alpha - \alpha\beta + \beta)} - \alpha e^{-u\mu^1} - e^{-\alpha u \mu^1} = 0. \tag{3.2}$$

- **Conditionally exponential independent distributions:**

$$(\alpha+1)\left(1 + \frac{u(\alpha - \alpha\beta + \beta)}{b}\right)^{-a} - \alpha\left(1 + \frac{u}{b}\right)^{-a} - \left(1 + \frac{u\alpha}{b}\right)^{-a} = 0. \tag{3.3}$$

Going through the limit $u \to \infty$ in equation (3.2) leads to $\beta \to 1$ while in equation (3.3) it leads to

$$\alpha + \alpha^{-a} - (\alpha+1)(\alpha - \alpha\beta + \beta)^{-a} = 0.$$

We deduce the following result for independent GPD.

Corollary 3.1. *If X^1 and X^2 are like in Section 2.5, with $\alpha > 1$, then the unique solution to (3.1) satisfies*

$$\beta(u) \underset{u \to \infty}{\longrightarrow} 1$$

where $\beta(u) = \frac{u^1}{u}$.

Proof. We apply Theorem 2.5 to X^1 and X^2 independent GPD with respective parameters $(\frac{1}{a}, \frac{b}{a})$ and $(\frac{1}{a}, \frac{b}{\alpha a})$, $a, b > 0$, $\alpha > 1$. We consider the quantities

$$\ell_1 = \frac{\mathbb{P}(X^1 \geq u^1,\ X^1 + X^2 \geq u)}{\mathbb{P}(X^1 \geq u^1)} \text{ and } \ell_2 = \frac{\mathbb{P}(X^2 \geq u^2,\ X^1 + X^2 \geq u)}{\mathbb{P}(X^1 \geq u^1)}.$$

If $u^1 + u^2 = u$ and (3.1) is satisfied then $\ell_1 = \ell_2$.
If $\beta(u) = \frac{u^1}{u}$ is bounded away from 0 and 1, then up to considering a sequence of real numbers u going to infinity, we may assume that

$$\frac{u^1}{u} \to \beta \text{ with } 0 < \beta < 1.$$

In that case, applying Theorem 2.5 gives

$$\lim_{u\to\infty} \ell_1 = \beta^a \text{ and } \lim_{u\to\infty} \ell_2 = \left(\frac{\beta}{\alpha}\right)^a$$

which is contradictory with the fact that $\ell_1 = \ell_2$.
So that, up to considering a sequence of real numbers u going to infinity, we have that either $\frac{u^1}{u} \to 0$ or $\frac{u^1}{u} \to 1$.
Let us assume that $\frac{u^1}{u} \to 0$. Following the lines of the proof of Theorem 2.5, we prove that

$$\lim_{u\to\infty} \frac{\mathbb{P}(X^1 \geq u^1,\ X^1 + X^2 \geq u)}{\overline{F}_{X^1}(u)} = 1 + \frac{1}{\alpha^a}$$

and

$$\lim_{u\to\infty} \frac{\mathbb{P}(X^2 \geq u^2,\ X^1 + X^2 \geq u)}{\overline{F}_{X^1}(u)} = \frac{1}{\alpha^a}.$$

This is in contradiction with the fact that the two above expressions are equal. We conclude that $\beta(u) \to 1$ as $u \to \infty$. □

We aim at finding a minimum $u^\star \in (\mathbb{R}_+)^d$ such that

$$u^\star = \mathbf{argmin}_{(u_1,\ldots,u_d)|u_1+\ldots+u_d=u} I(u_1, \ldots, u_d), \tag{3.4}$$

where I stands for one of the following indicators I_1, I_2 or J. We propose in [4] to solve this problem by using stochastic algorithms with a gradient descent method. This algorithm used for the estimation of the optimal allocation related to the *orange orea* can be easily adapted to the estimation of allocation minimizing the *violet area.* The keystone for the proof of convergence of this algorithm relies in the strong convexity of the auxiliary function used in the mirror algorithm, in the properties of the Legendre transform and in martingales techniques. The *violet area* as well as the *orange area* and the *stopped orange area* satisfy the hypothesis of convergence of the algorithm. We briefly present this stochastic algorithm in the next subsection and refer to [4] for more complete details. It is a Kiefer-Wolfowitz version of the stochastic mirror algorithm, which is particularly performant in high dimension (that is when d is quite large).

3.1. **Rapid description of the algorithm.** The stochastic mirror algorithm estimates the extremum of a function for which we can only observe a noisy version of its gradient ψ. From two step size sequences (β_n) and (γ_n), the algorithm consists in doing the gradient descent in the dual space in the following way.

Algorithm 3.2.

▷ *Initialization:*

$\xi_0 = 0$ *is in the dual space* E^* *of* $\mathbb{R}^d$*,*

$\chi_0 \in C := \{x \in \mathbb{R}^d,\ x^i \geq 0,\ x^1 + \cdots + x^d = u\}$

▷ *Update: for* $n = 1$ *to* N *do*

- $\xi_n = \xi_{n-1} - \gamma_n \psi(\chi_{n-1})$
- $\chi_n = \beta_n \ln \left(\frac{1}{d} \sum_{k=1}^{d} \exp\left(\frac{u.\xi_n^k}{\beta_n} \right) \right).$

▷ *Output:*

$$S^N = \frac{\sum_{n=1}^{N} \gamma_n \chi_{n-1}}{\sum_{n=1}^{N} \gamma_n}.$$

The function ψ is the discrete gradient of the vector $\mathcal{I} \in \mathbb{R}^d$ whose k^{th} coordinate is

$$\mathcal{I}^k = \left(\sum_{j=1}^{p} g_k(R_j^k) \mathbb{1}_{\{R_j^k < 0\}} \mathbb{1}_{\{\sum_{i=1}^d R_j^i \geq 0\}} \right)$$

in order to find the minimum of I_2, or of the vector $\mathcal{I}_S \in \mathbb{R}^d$ whose k^{th} coordinate is

$$\mathcal{I}_S^k = \left(\sum_{j=1}^{p \wedge \tau} g_k(R_j^k) \mathbb{1}_{\{R_j^k < 0\}} \mathbb{1}_{\{\sum_{i=1}^d R_j^i \geq 0\}} \right)$$

in order to find the minimum of I_2 or of the vector $\mathcal{J} \in \mathbb{R}^d$ whose k^{th} coordinate is

$$\mathcal{J}^k = \left(\sum_{j=1}^{p} g_k(R_j^k) \mathbb{1}_{\{R_j^k < 0\}} \mathbb{1}_{\{\sum_{i=1}^d R_j^i \leq 0\}} \right),$$

We refer to [4] for the suitable calibration of the sequences (β_n) and (γ_n) and for the proof of the consistency of the estimator S^N of u^*. The procedure may be summarized in Figure 2. It illustrates the fact that the gradient descent is done in the dual space and then pushed in the constraints space via the *transformation* of the sequence (ξ_n) to (χ_n).

4. Simulations for $p = 1$, $d = 2$

We shall perform simulations for $p = 1$ and $d = 2$ for the models described above (exponential independent = IE, conditionally exponential= CIE, independent GPD = IGPD). Recall that the IGPD and CIE models have the same margins. For the IE and CIE models, we compare the simulated results with the theoretical ones. The simulations are done for the *non crisis indicator* (*orange area*) as well as for the *crisis indicator* (violet area). Then, we have performed simulations in higher dimension ($d > 2$).

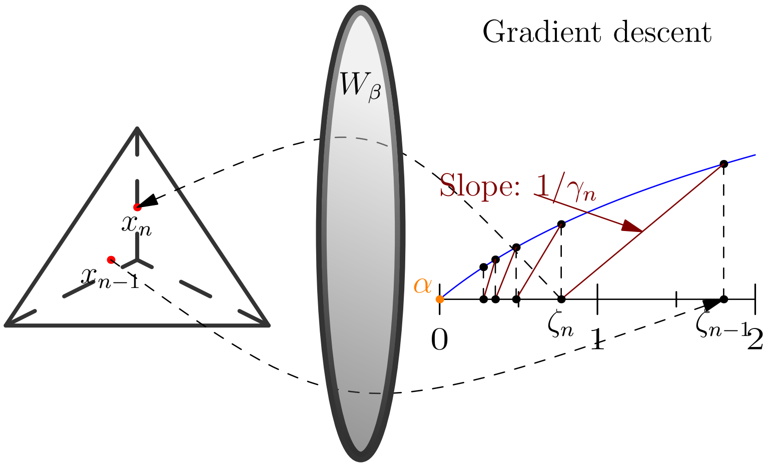

FIGURE 2. Illustration of the mirror algorithm.

4.1. **A comparison of our three models.** We have chosen $\mu^1 = \frac{1}{20}$ for the IE model, $a = 3$ and $b = 60$ for the CIE model and $\xi = \frac{1}{a}$, $\sigma = \frac{b}{a}$ for the IGPD model. In order to get the estimation of the minimum, we have performed 10 times the stochastic algorithm on data of length 15 000. The mean and the standard deviation over the 10 estimations are given below. For $\alpha = 5$ and $\alpha = 10$, we have taken $u = 50$, we compare with the theoretical value using the relative mean squared error (rmse) and the mean squared error (mse) for the IE and CIE models. Recall that if $\widehat{u}^i_\ell$ is the estimated value of u^i on the ℓ^{th} sample, then

$$rmse(u^i) = \frac{1}{\ell}\sum_{j=1}^{\ell}\left(\frac{\widehat{u}^i_\ell - u^i}{u^i}\right)^2,$$

and

$$mse(u^i) = \frac{1}{\ell}\sum_{j=1}^{\ell}\left(\widehat{u}^i_\ell - u^i\right)^2.$$

4.2. **The indicator I_2: the *orange area*.** Table 1 shows that, for the three models, the more the first branch is risky, the more we allocate capital to it. In the table above, the values of u are (relatively) small, we conclude that the asymptotic has not been reached.

4.3. **The indicator J: the *violet area*.** Table 2 shows the results of simulations for the *violet area*. The fact that the *rmse* and *mse* are larger than in the case of the *orange area* is due to the fact that less data satisfy the condition $\sum_{k=1}^{d} R_j^k \leq 0$ and thus the estimation is less accurate.

This simulation section should be used as a benchmark to calibrate the parameters of the algorithm. It shows that the algorithm perform well. Section 5 provides simulation in higher dimension; the impact of dependencies will be emphasized.

	$\alpha = 5$								
	IE model			CIE model			IGPD model		
	$\widehat{u}^1$	$\widehat{u}^2$	β	$\widehat{u}^1$	$\widehat{u}^2$	β	$\widehat{u}^1$	$\widehat{u}^2$	β
mean	38.37	11.63	0.767	36.8	13.2	0.735	35.83	14.17	0.717
sd dev	0.085	0.085	...	0.115	0.115	...	0.133	0.133	...
th.	38.46	11.54	0.769	36.84	13.16	0.737	not available		
$\sqrt{\text{rmse}}$	0.0032	0.0105	...	0.0032	0.009	...	not available		
$\sqrt{\text{mse}}$	0.121	0.121	...	0.119	0.119	...	not available		
	$\alpha = 10$								
mean	42.64	7.36	0.852	40.95	9.05	0.819	40.05	9.48	0.801
sd dev	0.161	0.161	...	0.138	0.138	...	0.164	0.164	...
th.	42.96	7.04	0.859	41.22	8.78	0.824	not available		
$\sqrt{\text{rmse}}$	0.0085	0.0516	...	0.0073	0.0342	...	not available		
$\sqrt{\text{mse}}$	0.364	0.363	...	0.301	0.3	...	not available		

TABLE 1. Simulations for the orange area.

	$\alpha = 5$								
	IE model			CIE model			IGPD model		
	$\widehat{u}^1$	$\widehat{u}^2$	β	$\widehat{u}^1$	$\widehat{u}^2$	β	$\widehat{u}^1$	$\widehat{u}^2$	β
mean	46.83	3.17	0.94	47.18	2.81	0.94	48.13	1.87	0.96
sd dev	0.35	0.35	...	0.136	0.136	...	0.114	0.114	...
th.	49.08	0.92	0.98	48.36	1.64	0.967	not available		
$\sqrt{\text{rmse}}$	0.046	2.49	...	0.025	0.722	...	not available		
$\sqrt{\text{mse}}$	2.28	2.28	...	1.19	1.19	...	not available		
	$\alpha = 10$								
mean	47.62	2.38	0.95	48.68	1.32	0.819	49.16	0.84	0.98
sd dev	0.43	0.43	...	0.083	0.083	...	0.063	0.063	...
th.	49.77	0.23	0.995	49.6	0.39	0.99	not available		
$\sqrt{\text{rmse}}$	0.044	9.7	...	0.019	2.32	...	not available		
$\sqrt{\text{mse}}$	2.19	2.19	...	0.92	0.92	...	not available		

TABLE 2. Simulations for the violet area.

5. Larger number of lines of business

We consider 10 lines, with different situations. We begin with some generalization of the previous models and then propose a model for which we have a block of 5 correlated lines of business, and a block of lines which are mutually independent.

5.1. **Generalizations of the previous models in dimension** 10.

- Independent exponential: $X_1^1 \rightsquigarrow \mathcal{E}(\mu^1)$, for $i = 2, \ldots, 7$, $X_1^i \rightsquigarrow \mathcal{E}(\alpha^1\mu^1)$ and for $i = 8, \ldots, 10$, $X_1^i \rightsquigarrow \mathcal{E}(\alpha^2\mu^1)$.
- Conditional exponential: $X_1^1 \rightsquigarrow \mathcal{E}(\Theta)$, for $i = 2, \ldots, 7$, $X_1^i \rightsquigarrow \mathcal{E}(\alpha^1\Theta)$ and for $i = 8, \ldots, 10$, $X_1^i \rightsquigarrow \mathcal{E}(\alpha^2\Theta)$, where $\Theta \rightsquigarrow \Gamma(a, b)$.
- Independent GPD: $X_1^1 \rightsquigarrow GPD(\frac{1}{a}, \frac{b}{a})$, for $i = 2, \ldots, 7$, $X_1^i \rightsquigarrow GPD(\frac{1}{a}, \frac{b}{\alpha_1 a})$ and for $i = 8, \ldots, 10$, $X_1^i \rightsquigarrow GPD(\frac{1}{a}, \frac{b}{\alpha_2 a})$.

The last two models have the same margins. We have chosen $\alpha^1 = 5$ and $\alpha^2 = 8$ and $u = 80$. We have performed our stochastic algorithm 10 times on data sets of length 20 000 for the *orange area*. The results are given in

	IE			**CIE**			**IGPD**		
	mean	sd dev.	$\frac{u^i}{u}$	mean	sd dev.	$\frac{u^i}{u}$	mean	sd dev.	$\frac{u^i}{u}$
u^1	27.04	0.095	0.338	25.68	0.194	0.321	22.38	0.178	0.28
u^2	6.7	0.031	0.084	6.85	0.05	0.086	7.17	0.044	0.09
u^3	6.69	0.039	0.084	6.83	0.083	0.085	7.26	0.06	0.091
u^4	6.69	0.037	0.084	6.87	0.054	0.086	7.24	0.054	0.09
u^5	6.7	0.051	0.084	6.88	0.079	0.086	7.18	0.061	0.09
u^6	6.71	0.078	0.084	6.85	0.066	0.086	7.21	0.087	0.09
u^7	6.72	0.021	0.084	6.86	0.063	0.086	7.19	0.081	0.09
u^8	4.23	0.026	0.053	4.41	0.028	0.055	4.79	0.034	0.06
u^9	4.26	0.032	0.053	4.4	0.057	0.055	4.77	0.059	0.06
u^{10}	4.26	0.028	0.053	4.38	0.057	0.055	4.81	0.036	0.06

TABLE 3. Simulations for Independent Exponentials, Correlated GPD and Independent GPD, in dimension 10 for the *orange area*.

Table 3. The same observations as in dimension 2 can be done. Moreover unsurprisingly, the allocation is identical for the branches 2 to 7 and for the branches 8 to 10.

5.2. **A block of correlated GPD and a block of independent GDP, in dimension** 10. In this section, we consider a block of 5 conditionally exponential variables and an independent block of 5 independent GPD variables, with the same margins as the 5 correlated GPD variables of the first block. We refer this model to the mixed model (MM).

- Conditional exponential: $X_1^1 \rightsquigarrow \mathcal{E}(\Theta)$ and for $i = 2, \dots, 5$, $X_1^i \rightsquigarrow \mathcal{E}(\alpha^1\Theta)$, where $\Theta \rightsquigarrow \Gamma(a, b)$.
- Independent GPD: $X_1^1 \rightsquigarrow GPD(\frac{1}{a}, \frac{b}{a})$ and for $i = 6, \dots, 10$, $X_1^i \rightsquigarrow GPD(\frac{1}{a}, \frac{b}{\alpha^1 a})$.

The MM model has to be compared with the IGPD and CIE models with the same margins. We have chosen $\alpha^1 = 5$ and $u = 80$. We have performed our stochastic algorithm 10 times on data sets of length 20 000 and present the results in Table 4. This example illustrates the fact that for the expected *orange area*, lines correlated to the most dangerous one receive less capital that lines with the same marginal distributions but independent from the riskiest line. This is in accordance with the basic example presented in the introduction. If line 1 is insolvent, then it is likely that the group is ruined (the aggregate reserves are likely to be negative). Consequently, a greater part of the insolvencies of lines 2 to 5 than the ones of lines 6 to 10 will not contribute to the *orange area* because the sum of reserves is negative. Once again, the *orange area* should not be used for economic capital allocation, as it encourages to take additional risks that are correlated to the main source of risk. Logically, the effect is reversed if one uses the expected *violet area*. In that case, a greater part of insolvencies of lines 6 to 10 will not contribute, and consequently the required capital for lines 2 to 5 will be higher than for lines 6 to 10, as in the TVaR allocation principle.

	MM			CIE			IGPD		
	mean	sd dev.	$\frac{u^i}{u}$	mean	sd dev.	$\frac{u^i}{u}$	mean	sd dev.	$\frac{u^i}{u}$
u^1	20.56	0.15	0.257	21.48	0.23	0.269	18.42	0.219	0.23
u^2	5.78	0.043	0.072	6.49	0.123	0.081	6.88	0.138	0.086
u^3	5.77	0.05	0.072	6.52	0.133	0.082	6.88	0.151	0.086
u^4	5.8	0.059	0.072	6.49	0.116	0.081	6.82	0.16	0.085
u^5	5.79	0.052	0.072	6.52	0.116	0.082	6.82	0.132	0.085
u^6	7.25	0.059	0.091	6.5	0.11	0.081	6.81	0.113	0.085
u^7	7.25	0.071	0.091	6.5	0.085	0.081	6.89	0.154	0.086
u^8	7.31	0.071	0.091	6.5	0.13	0.081	6.82	0.14	0.085
u^9	7.25	0.066	0.091	6.49	0.121	0.081	6.82	0.143	0.085
u^{10}	7.26	0.078	0.091	6.49	0.121	0.081	6.83	0.153	0.085

TABLE 4. Simulations for a block of correlated GPD and a block of independent GDP, in dimension 10 for the *orange area*.

In Table 5 stand the results for the same models as above, in the case of the *violet area* (indicator J). We have performed 10 simulations of length 23 000.

	MM			CIE			IGPD		
	mean	sd dev.	$\frac{u^i}{u}$	mean	sd dev.	$\frac{u^i}{u}$	mean	sd dev.	$\frac{u^i}{u}$
u^1	41.4	0.23	0.52	37.68	0.31	0.47	41.99	0.254	0.52
u^2	5.2	0.07	0.065	4.68	0.068	0.058	4.19	0.044	0.052
u^3	5.19	0.08	0.065	4.72	0.076	0.059	4.23	0.046	0.053
u^4	5.18	0.07	0.065	4.67	0.069	0.058	4.2	0.073	0.053
u^5	5.16	0.07	0.065	4.73	0.063	0.059	4.2	0.094	0.052
u^6	3.57	0.08	0.045	4.68	0.053	0.059	4.22	0.048	0.053
u^7	3.56	0.06	0.044	4.72	0.094	0.059	4.23	0.069	0.053
u^8	3.58	0.05	0.045	4.69	0.046	0.058	4.24	0.07	0.053
u^9	3.56	0.05	0.045	4.69	0.04	0.059	4.25	0.064	0.053
u^{10}	3.59	0.06	0.045	4.72	0.069	0.059	4.24	0.051	0.053

TABLE 5. Simulations for a block of correlated GPD and a block of independent GDP, in dimension 10 for the *violet area*.

As expected, the optimal allocation with respect to the *violet area* leads to penalize all the branches correlated to the more risky one; the independent branches are less penalized.

6. Time horizon $p > 1$, d lines of business

We finish our simulation series with several multi-periodic simulations. We have performed simulations only for models with independence in time. As already mentioned, for multi-periodic cases, the premium c has to be taken into account. We set the safety loading[c] to 5%. We have performed 10 simulations of length 15 000, for the three following models, with $\mu_1 = \frac{1}{20}$, $\alpha = 5$, $a = 3$, $b = 60$:

[c]That means that the premium c^k of the kth branch is $1.05 \times \mathbb{E}(X_1^k)$.

- Independent exponential: $X_j^1 \rightsquigarrow \mathcal{E}(\mu^1)$, for $i = 2, \ldots, d$, $X_j^i \rightsquigarrow \mathcal{E}(\alpha\mu^1)$, $j = 1, \ldots, p$;
- Conditional exponential (Correlated GPD): $X_j^1 \rightsquigarrow \mathcal{E}(\Theta)$, for $i = 2, \ldots, d$, $X_j^i \rightsquigarrow \mathcal{E}(\alpha\Theta)$, where $\Theta \rightsquigarrow \Gamma(a, b)$, $j = 1, \ldots, p$;
- Independent GPD: $X_j^1 \rightsquigarrow GPD(\frac{1}{a}, \frac{b}{a})$, for $i = 2, \ldots, d$, $X_j^i \rightsquigarrow GPD(\frac{1}{a}, \frac{b}{\alpha a})$, $j = 1, \ldots, p$.

The vectors $(X_j^1, \ldots, X_j^d)$ are independant, that is, there is no time dependencies. The simulations are done for the *orange area* (indicator I_2 in Table 6) in dimension $d = 3$ and for $u = 30$.

	IE			**CIE**			**IGPD**		
	mean	sd dev.	$\frac{u^i}{u}$	mean	sd dev.	$\frac{u^i}{u}$	mean	sd dev.	$\frac{u^i}{u}$
	$p = 2$								
u^1	17.06	0.11	0.586	13.67	0.26	0.456	11.87	0.19	0.396
u^2	6.49	0.1	0.216	8.16	0.13	0.272	9.1	0.22	0.303
u^3	6.45	0.07	0.215	8.16	0.24	0.272	9.03	0.18	0.301
	$p = 3$								
u^1	16.5	0.1	0.55	12.85	0.18	0.428	10.52	0.22	0.351
u^2	6.75	0.08	0.225	8.56	0.15	0.285	9.76	0.15	0.325
u^3	6.74	0.11	0.225	8.58	0.1	0.286	9.72	0.15	0.324
	$p = 4$								
u^1	15.99	0.12	0.533	11.8	0.16	0.393	9.48	0.16	0.316
u^2	7.01	0.08	0.234	9.1	0.12	0.303	10.27	0.16	0.342
u^3	6.99	0.09	0.233	9.1	0.08	0.303	10.25	0.11	0.342

TABLE 6. Multi-periodic simulations for Independent Exponentials, Correlated GPD and Independent GPD, in dimension 3 for the *orange area* with $u = 30$, $p = 2, 3, 4$.

6.1. **The *orange area*, the I_2 indicator.** These multi-periodic simulations show the same kind of behavior as for $p = 1$. Nevertheless, we remark that for the IGPD model, with $p = 4$, the more risky branch becomes less allocated. This is a quite surprising result which may be explained by the fact that $u = 30$ in this context is quite small (recall that the premiums are added at each time $n = 1, \ldots, p$). For larger values of u, this phenomenon does not appear anymore as it can be seen in Table 7 ($u = 80$). As in

	IE			**CIE**			**IGPD**		
	mean	sd dev.	$\frac{u^i}{u}$	mean	sd dev.	$\frac{u^i}{u}$	mean	sd dev.	$\frac{u^i}{u}$
u^1	53.77	0.17	0.672	46.02	0.24	0.575	43.98	0.29	0.55
u^2	13.12	0.12	0.164	16.99	0.11	0.212	18.03	0.26	0.225
u^3	13.11	0.09	0.164	16.98	0.23	0.212	17.99	0.21	0.225

TABLE 7. Multi-periodic simulations for Independent Exponentials, Correlated GPD and Independent GPD, in dimension 3 for the *orange area* with $u = 80$, $p = 4$.

dimension 10, the case of mixed models is more interesting. We have done a last simulation with $p = 4$, $d = 5$, $u = 80$, resumed in Table 8. The first 3 lines are conditionally exponential and the last two lines are independent GPD. More precisely, for $j = 1, \dots, p$,

- Conditional exponential: $X_j^1 \rightsquigarrow \mathcal{E}(\Theta)$ and for $i = 2, 3$, $X_j^i \rightsquigarrow \mathcal{E}(\alpha\Theta)$, where $\Theta \rightsquigarrow \Gamma(a, b)$.
- Independent GPD: $X_j^1 \rightsquigarrow GPD(\frac{1}{a}, \frac{b}{a})$ and for $i = 4, 5, 6$, $X_j^i \rightsquigarrow GPD(\frac{1}{a}, \frac{b}{\alpha a})$,

with $\alpha = 5$, $a = 3$ and $b = 60$.

	mean	sd dev.	$\frac{u^i}{u}$
u^1	31.33	0.36	0.392
u^2	10.63	0.16	0.133
u^3	10.6	0.24	0.133
u^4	13.74	0.27	0.172
u^5	13.7	0.27	0.171

TABLE 8. Multiperiodic simulations for a block of correlated GPD and a block of independent GDP (mixed model MM), in dimension 5 with $u = 80$, $p = 4$.

It has to be compared with the correlated Pareto and independent GPD models with the same margins. This is summarized in Table 9. As for $p = 1$,

	IE		**CIE**		**IGPD**	
	mean	sd dev.	mean	sd dev.	mean	sd dev.
u^1	41.1	0.23	32.68	0.28	30.1	0.31
u^2	9.69	0.09	11.86	0.14	12.49	0.17
u^3	9.71	0.08	11.78	0.24	12.51	0.23
u^4	9.78	0.12	11.83	0.15	12.4	0.14
u^5	9.72	0.09	11.85	0.12	12.5	0.2

TABLE 9. Multiperiodic simulations for Independent Exponentials, Correlated GPD and Independent GPD, in dimension 5 with $u = 80$, $p = 4$.

we remark that branches correlated with the most risky one have less capital allocation. So that, even for multi-periodic settings, the orange area should be useful to allocate some free (or investment) capital but not for solvency capital.

6.2. **The stopped *orange area*, the I_1 indicator.** It is expected that the behavior of the stopped and the non stopped *orange area* are quite similar because in both cases, the indicator does not take into account the event which leads to global insolvency. If there is only one period ($p = 1$), there is no difference between the I_2 and the I_1 indicators. In order to compare the optimization with the I_1 indicator with respect to the I_2, we have performed simulations (see Table 10) with $p = 5$ and $d = 5$ for $u = 80$ on the same models as above (independent exponential, conditionally independent exponential, independent GPD, mixed GPD). We remark that the only notable difference is observed for the mixed model.

	IE		CIE		IGPD		MM	
	mean	sd dev.	mean	sd dev.	mean	sd dev.	mean	sd dev.
u^1	41.13	0.22	32.9	0.3	30.12	0.22	27.34	0.38
u^2	9.73	0.1	11.79	0.15	12.44	0.14	11.25	0.22
u^3	9.72	0.08	11.79	0.15	12.51	0.21	11.15	0.15
u^4	9.72	0.1	11.79	0.19	12.52	0.2	15.1	0.26
u^5	9.71	0.12	11.72	0.19	12.42	0.17	15.16	0.23

TABLE 10. Multi-periodic simulations for Independent Exponentials, Correlated GPD and Independent GPD, in dimension 5 for the *stopped orange area* with $u = 80$, $p = 4$.

6.3. **The *violet area*, the J indicator.** As already noticed, for the *violet area*, more data is needed because less of the aggregate loss go over u. The simulation below are done on data of length 16 000, 10 times. We have considered first Independent Exponentials, Correlated GPD and Independent GPD, in dimension 5 with $u = 80$, $p = 3$, 4, 5 (see Table 11).

	IE			CIE			IGPD		
	mean	sd dev.	$\frac{u^i}{u}$	mean	sd dev.	$\frac{u^i}{u}$	mean	sd dev.	$\frac{u^i}{u}$
	$p = 3$								
u^1	45.35	2.91	0.567	70.2	1.2	0.877	77.81	0.7	0.972
u^2	8.54	0.82	0.107	2.42	0.34	0.03	0.53	0.19	0.007
u^3	8.72	0.59	0.109	2.52	0.39	0.032	0.56	0.21	0.007
u^4	8.65	0.92	0.108	2.34	0.29	0.029	0.54	0.19	0.007
u^5	8.74	0.68	0.109	2.52	0.35	0.032	0.56	0.17	0.007
	$p = 4$								
u^1	62.63	2.37	0.78	73.63	0.326	0.92	79.6	0.157	0.995
u^2	4.31	0.6	0.053	1.62	0.119	0.02	0.1	0.041	0.001
u^3	4.34	0.6	0.054	1.52	0.194	0.019	0.11	0.042	0.001
u^4	4.41	0.68	0.055	1.58	0.327	0.02	0.09	0.033	0.001
u^5	4.3	0.59	0.054	1.65	0.203	0.021	0.11	0.045	0.002
	$p = 5$								
u^1	55.12	3.26	0.689	71.61	0.79	0.895	78.41	0.52	0.98
u^2	6.23	0.83	0.078	2.06	0.3	0.026	0.37	0.12	0.005
u^3	6.29	0.95	0.079	1.96	0.41	0.024	0.41	0.15	0.005
u^4	6.18	0.91	0.077	2.24	0.24	0.028	0.42	0.17	0.005
u^5	6.18	0.89	0.077	2.13	0.17	0.027	0.4	0.14	0.005

TABLE 11. Multi-periodic simulations for Independent Exponentials, Correlated GPD and Independent GPD, in dimension 5 for the *violet area* with $u = 80$, $p = 3$, 4, 5.

We have also performed simulations (see Table 12) for the mixed GPD model (the first 3 lines are conditionally exponential and the last two lines are independent GPD with the parameters which are the same as above). As for $p = 1$, we remark that branches correlated with the most risky one have more capital allocation. So that, even for multi-periodic settings, the

	$p=3$			$p=4$			$p=5$		
	mean	sd dev.	$\frac{u^i}{u}$	mean	sd dev.	$\frac{u^i}{u}$	mean	sd dev.	$\frac{u^i}{u}$
u^1	74.74	0.76	0.934	77.64	0.24	0.971	76.3	0.85	0.954
u^2	2.32	0.34	0.029	1.11	0.11	0.014	1.74	0.36	0.022
u^3	2.18	0.47	0.027	1.1	0.23	0.014	1.56	0.49	0.019
u^4	0.38	0.12	0.005	0.066	0.02	0.001	0.21	0.07	0.003
u^5	0.37	0.09	0.005	0.072	0.02	0.001	0.2	0.08	0.003

TABLE 12. Multi-periodic simulations for a block of correlated GPD and a block of independent GDP (mixed model MM), in dimension 5 for the *violet area* with $u = 80$, $p = 3$, 4, 5.

violet area should be useful to allocate solvency capital (compare the CIE, IGPD and MM models with the same margins).

7. Conclusion

In this article, we have presented some results for the allocation of capital with respect to the minimization of some risk indicators. In dimension $d = 2$ for the one period ($p = 1$) case, explicit formulas and some asymptotics are given for specific models, a simulation study is done, it allows to benchmark the parameters of the optimization algorithm. Then, simulations are done in higher dimension and for several periods ($p > 1$). It emphasizes the fact that minimization driven allocations require to choose the right risk indicator. As an example, we would recommend to use the orange area for free capital allocation, while the violet area is more consistent for economic capital allocation. Also, the impact on the allocation of some time dependency should be studied.

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