Elementary partial differential equations: correction of Midterm 1

Exercise 1

The proposed three PDE are linear and first-order. A natural means to solve them is then the method of characteristics.

(1) We consider the PDE:

$$-\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} + xu = 0. \tag{1}$$

Its characteristic curves $s \mapsto (x(s), y(s))$ fulfill the following ODE:

$$\left\{ \begin{array}{ll} x'(s) = & -1 \\ y'(s) = & 2 \end{array} \right. , \tag{2}$$

which gives,

$$\left\{ \begin{array}{ll} x(s) = & -s \\ y(s) = & 2s + c \end{array} \right. ,$$

where c is an arbitrary constant (as usual, strictly speaking, two arbitrary constants should stem from the integration of (2), one of which disappearing by operating a reparameterization of the curves; see the correction of Homework 1).

Now, consider a given characteristic curve (i.e. c is fixed), and let $z(s) := u(x(s), y(s))$ be the value of the solution to the PDE along this curve. Differentiating in the definition of $z(s)$ yields a subsequent ODE:

$$\begin{array}{rcl} z'(s) & = & x'(s)\frac{\partial u}{\partial x}(x(s), y(s)) + y'(s)\frac{\partial u}{\partial y}(x(s), y(s)) \\ & = & -\frac{\partial u}{\partial x}(x(s), y(s)) + 2\frac{\partial u}{\partial y}(x(s), y(s)) \\ & = & -x(s)z(s) \\ & = & sz(s) \end{array} ,$$

where the first line stems from the use of chain rule, the second line from (2), and the third line is nothing but (1) evaluated at point $(x(s), y(s))$. Solving this ODE then yields the existence of a constant D such that:

$$z(s) = De^{\frac{s^2}{2}}.$$

Now, we come to the final resolution of our PDE (1): the constant D in the expression of $z(s)$ depends on the considered characteristic curve (i.e. on c), and we now rewrite:

$$u(x(s), y(s)) = z(s) = f(c)e^{\frac{s^2}{2}},$$

where f is an arbitrary differentiable function. Eventually, using the formulae () which express the characteristic curve c and the parameter s on this curve of an arbitrary point (x, y) in the plane, we get:

$$u(x, y) = f(y + 2x)e^{\frac{x^2}{2}},$$

where f is an arbitrary function. Now, you only have to check this result (which you should do *systematically*)!

If now $u(x, 0) = 2xe^{\frac{x^2}{2}}$, then $f(2x)e^{\frac{x^2}{2}} = 2xe^{\frac{x^2}{2}}$ and $f(x) = x$. Eventually:

$$u(x, y) = (y + 2x)e^{\frac{x^2}{2}}.$$

(2) We now turn to the PDE

$$2x\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0.$$

Its characteristic curves $s \mapsto (x(s), y(s))$ fulfill the following ODE:

$$\left\{ \begin{array}{ll} x'(s) = & 2x(s) \\ y'(s) = & 1 \end{array} \right. ,$$

which gives,

$$\begin{cases} x(s) = & ce^{2s} \\ y(s) = & s \end{cases},$$

where c is an arbitrary constant.

Considering a fixed characteristic curve (i.e. c is fixed), the value function $z(s) = u(x(s), y(s))$ along this curve satisfies:

$$z'(s) = 0.$$

Consequently, the solution to our PDE reads:

$$u(x(s), y(s)) = f(c),$$

where f is an arbitrary differentiable function. Expressing this relation in terms of x and y only yields:

$$u(x, y) = f(xe^{-2y}).$$

Eventually, using the 'boundary condition' $u(x, 0) = \sin(x)$, we have $f(x) = \sin(x)$ and:

$$u(x, y) = \sin(xe^{-2y}).$$

(3) Let us finally deal with the PDE

$$(1 + x^2)\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} - u = e^y. \tag{3}$$

Its characteristic curves $s \mapsto (x(s), y(s))$ fulfill the following ODE:

$$\begin{cases} x'(s) = & 1 + x^2(s) \\ y'(s) = & 1 \end{cases},$$

which gives,

$$\begin{cases} \arctan(x(s)) & = & s + c \\ y(s) & = & s \end{cases},$$

where c is an arbitrary constant.

Considering a fixed characteristic curve (i.e. c is fixed), the value function $z(s) = u(x(s), y(s))$ along this curve satisfies:

$$\begin{array}{rcl} z'(s) & = & x'(s)\frac{\partial u}{\partial x}(x(s), y(s)) + y'(s)\frac{\partial u}{\partial y}(x(s), y(s)) \\ & = & (1 + x^2(s))\frac{\partial u}{\partial x}(x(s), y(s)) + \frac{\partial u}{\partial y}(x(s), y(s)) \\ & = & z(s) + e^s \end{array}.$$

We now end up with an ODE:

$$z'(s) - z(s) = e^s. \tag{4}$$

The solution $z_0(s)$ of the corresponding homogeneous ODE is simply:

$$z_0(s) = De^s,$$

where D is an arbitrary constant. Now, we apply the method of variation of constant to solve the inhomogeneous ODE (4), and search for its solution under the form $z(s) = D(s)e^s$, where $D(s)$ is a function to be found. Plugging this expression into (4) yields:

$$D'(s)e^s = e^s,$$

and $D'(s) = 1$. Thus, there is a constant E such that:

$$z(s) = se^s + Ee^s.$$

Now, we return to our PDE (3). The constant E appearing above as usual depends on the particular characteristic curve considered in the analysis (hence, it is actually a function $f(c)$). Expressing the value function in terms of x and y instead of s and c gives:

$$u(x, y) = ye^y + f(y - \arctan(x))e^y,$$

Exercise 2

(1) (a) has determinant $\Delta = 0.0 - (\frac{5}{2})^2 = -\frac{25}{4}$ and is therefore hyperbolic. (b) has determinant $\Delta = 1.2 - (1)^2 = 1$ and is elliptic. Eventually, (c) has determinant $\Delta = 4.0 - 0^2 = 0$ and is parabolic.

(2) The determinant of the PDE

$$x\frac{\partial^2 u}{\partial x^2} + (x+y)\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} = 0 \tag{5}$$

depends on the considered point since its coefficients are not constant. It reads:

$$\begin{aligned} \Delta(x,y) &= xy - \left(\tfrac{x+y}{2}\right)^2 \\ &= \tfrac{4xy-(x^2+y^2+2xy)}{4} \\ &= \tfrac{-(x^2+y^2-2xy)}{4} \\ &= -\left(\tfrac{x-y}{2}\right)^2 \end{aligned}.$$

This last expression is:

- null if $x = y$, in which case (5) is parabolic
- negative if $x \neq y$, in which case (5) is hyperbolic.

The situation is then as depicted in Figure 1.

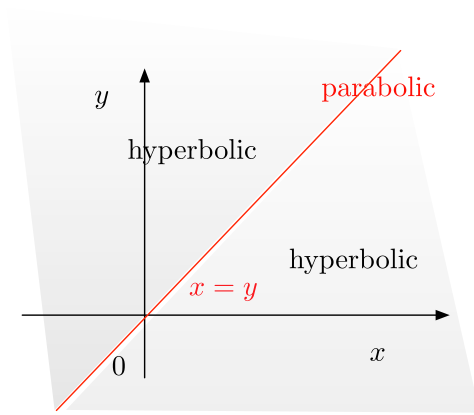

FIGURE 1. Sketch of the regions where the PDE (5) is elliptic, parabolic or hyperbolic.

(3) We consider the second-order linear PDE:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 3\frac{\partial u}{\partial x} - 4\frac{\partial u}{\partial y} + 25u = 0. \tag{6}$$

(a): This simply makes use of the formula for differentiating a product of two functions. We have:

$$\frac{\partial u}{\partial x}(x,y) = ae^{ax+by}v(x,y) + e^{ax+by}\frac{\partial v}{\partial x}(x,y), \quad \frac{\partial u}{\partial y}(x,y) = be^{ax+by}v(x,y) + e^{ax+by}\frac{\partial v}{\partial y}(x,y),$$

and for the second-order derivatives:

$$\frac{\partial^2 u}{\partial x^2}(x,y) = a^2e^{ax+by}v(x,y) + 2ae^{ax+by}\frac{\partial v}{\partial x}(x,y) + e^{ax+by}\frac{\partial^2 v}{\partial x^2}(x,y),$$
$$\frac{\partial^2 u}{\partial y^2}(x,y) = b^2e^{ax+by}v(x,y) + 2be^{ax+by}\frac{\partial v}{\partial y}(x,y) + e^{ax+by}\frac{\partial^2 v}{\partial y^2}(x,y).$$

(b): Plugging the above expressions in (6) yields:

$$e^{ax+by}\frac{\partial^2 v}{\partial x^2} + e^{ax+by}\frac{\partial^2 v}{\partial y^2} + (2a+3)\,e^{ax+by}\frac{\partial v}{\partial x} + (2b-4)\,e^{ax+by}\frac{\partial v}{\partial y} + \left(a^2+b^2+3a-4b+25\right)e^{ax+by}v = 0,$$

i.e.:

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + (2a+3)\frac{\partial v}{\partial x} + (2b-4)\frac{\partial v}{\partial y} + \left(a^2+b^2+3a-4b+25\right)v = 0.$$

Consequently, for v to satisfy a PDE of the form:

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + cv = 0,$$

(i.e. the first-order term have disappeared, and the second-order ones are unchanged), one must chose $a = -\frac{3}{2}$ and $b = 2$. In this situation, the 0^{th} order coefficient c equals:

$$c = a^2 + b^2 + 3a - 4b + 25 = \frac{75}{4}.$$

Exercise 3

(1) $u(t,x)$ solves the following system, composed o the heat equation, and supplemented with initial conditions, and Neumann boundary conditions:

$$\begin{gathered}\forall t>0,\ \forall x\in(0,L),\ c\rho\frac{\partial u}{\partial t}(t,x) - \kappa\frac{\partial^2 u}{\partial x^2} = 0,\\ \forall t>0,\ -\kappa\frac{\partial u}{\partial x}(t,0) = \alpha,\ -\kappa\frac{\partial u}{\partial x}(t,L) = \beta,\\ \forall x\in(0,L),\ u(0,x) = \phi(x).\end{gathered}$$

$\alpha = -\kappa\frac{\partial u}{\partial x}(t,0)$ is the heat flux crossing through the 'surface' $x=0$, from the left to the right, at time t. It is therefore a flux going *in* the rod. On the contrary, $\beta = -\kappa\frac{\partial u}{\partial x}(t,L)$ is the heat flux crossing through the 'surface' $x=L$, from the left to the right, at time t, and corresponds to an energy *leaving* the rod.

(2). Since the cross- sectional area A is constant, the total heat $H(t)$ reads:

$$H(t) = A\int_0^L c\rho u(t,x)\,dx.$$

(3) Let us compute:

$$\begin{aligned} H'(t) &= A\frac{d}{dt}\left(\int_0^L c\rho u(t,x)\,dx\right)\\ &= A\int_0^L c\rho\frac{\partial u}{\partial t}(t,x)\,dx\\ &= A\int_0^L \kappa\frac{\partial^2 u}{\partial x^2}(t,x)\,dx\\ &= A\left(\kappa\frac{\partial u}{\partial x}(t,L) - \kappa\frac{\partial u}{\partial x}(t,0)\right)\\ &= A(\alpha-\beta),\end{aligned}$$

where the second line follows from the theorem of differentiation under the integral sign, and the third line is a consequence of the fact that u solves the heat equation. As a consequence, one has:

$$\forall t\geq 0,\ H(t) = A(\alpha-\beta)t + H(0). \tag{7}$$

(4) An equilibrium state to this equation does not exist under any circumstances. Indeed, relation (7) shows that the heat energy $H(t)$ contained in the rod goes tremendously large in time (i.e. $H(t)\to\pm\infty$), which prevents the existence of such a state... *unless* the 'exploding' term $A(\alpha-\beta)t$ vanishes, which is only possible provided $\alpha=\beta$. This can be easily understood, since the total heat energy *received* by the rod from the outside at time t is α (see the first question), and the total energy *leaving* the rod at time t is β. An equilibrium state can exist only if those two terms compensate one another, i.e. $\alpha=\beta$.

Exercise 4

(1) See the lectures (or the textbook)!

(2) This is a direct consequence of the maximum and of the minimum principles. We know that:

$$\forall x \in [0,1],\ \ u(0,x) = 4x(1-x) \leq 1,$$

and, as far as the boundary conditions are concerned:

$$\forall t > 0,\ \ u(t,0) = 0 \leq 1,\ \ u(t,1) = 0 \leq 1.$$

The maximum principle allows to conclude that:

$$\forall t > 0,\ \ \forall x \in [0,1],\ \ u(t,x) \leq 1.$$

In the very same way, using the minimum principle yields:

$$\forall t > 0,\ \ \forall x \in [0,1],\ \ 0 \leq u(t,x).$$

(3) As suggested by the hint, denote as $v(t,x) = u(t,1-x)$. We have, by the chain rule:

$$\frac{\partial v}{\partial t}(t,x) = \frac{\partial u}{\partial t}(t,1-x),$$

and:

$$\frac{\partial v}{\partial x}(t,x) = -\frac{\partial u}{\partial x}(t,1-x), \frac{\partial^2 v}{\partial x^2}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,1-x).$$

As a consequence:

$$\left(\frac{\partial v}{\partial t} - \kappa \frac{\partial^2 v}{\partial x^2}\right)(t,x) = \left(\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2}\right)(t,1-x) = 0.$$

What's more, we also have:

$$\forall t > 0,\ \ v(t,0) = u(t,1) = 0,\ \ v(t,1) = u(t,0) = 0,$$

and

$$\forall x \in [0,1],\ \ v(0,x) = u(0,1-x) = 4(1-x)(1-(1-x)) = 4x(1-x) = u(0,x).$$

All in all, v and u are two solutions of the heat equation $\frac{\partial u}{\partial t} - \kappa \frac{\partial u}{\partial x^2} = 0$, with the *same* homogeneous Dirichlet boundary conditions, and the *same* initial conditions $u(0,x) = 4x(1-x)$. As we have seen during the lectures, the solution to such a system is unique. Consequently:

$$\forall t > 0,\ \ \forall x \in [0,1],\ \ u(t,x) = v(t,x) = u(t,1-x).$$

(4) This is a simple computation, as we have seen several times during the lectures:

$$\begin{array}{rcl} E'(t) & = & \frac{1}{2}\frac{d}{dt}\left(\int_0^L u^2(t,x)\,dx\right) \\ & = & \int_0^L u(t,x)\frac{\partial u}{\partial t}(t,x)\,dx \\ & = & \int_0^L \kappa u(t,x)\frac{\partial^2 u}{\partial x^2}(t,x)\,dx \\ & = & \left[u(t,x)\frac{\partial u}{\partial x}(t,x)\right]_0^L - \kappa\int_0^L \left(\frac{\partial u}{\partial x}\right)^2(t,x)\,dx \\ & = & -\kappa\int_0^L \left(\frac{\partial u}{\partial x}\right)^2(t,x)\,dx \\ & \leq & 0 \end{array},$$

where the second line follows from the theorem of differentiation under the integral sign, the third line is a consequence of the fact that u solves the heat equation, the fourth line stems from an integration by parts, and the fifth line is obtained by using the information about the homogeneous Dirichlet boundary conditions.

Exercise 5

(1) With the notations proposed in the midterm, the *d'Alembert formula* for the solution to the wave equation is:

$$\forall t > 0,\ \forall x \in \mathbb{R},\ u(t,x) = \frac{1}{2}\left(\phi(x-ct) + \phi(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds.$$

(2) The only thing to do here is to plug the supplied data in the above formula:

$$\begin{aligned} u(t,x) &= \tfrac{1}{2}\left(\cos(x-ct) + \cos(x+ct)\right) + \tfrac{1}{2c}\textstyle\int_{x-ct}^{x+ct} 0\, ds. \\ &= \tfrac{1}{2}\left(\cos(x)\cos(ct) + \sin(x)\sin(ct) + (\cos(x)\cos(ct) - \sin(x)\sin(ct))\right) \\ &= \cos(x)\cos(ct), \end{aligned}$$

as desired.

(3) I am pretty sure you will manage to draw the graph of the 'cos' function by yourselves!