

Elementary partial differential equations: 1st midterm

This midterm is purposely too long for you to complete within the allowed time, and of course, it is not assumed that you cover all the exercises to get the maximal grade. The indicated grading policy is only provisional, and aimed at illustrating the relative weights of the different exercises.

Exercise 1 (6pts)

- (1) Find the solution to the PDE

$$-\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} + xu = 0$$

of a function $u \equiv u(x, y)$ of two variables, such that $u(x, 0) = 2xe^{\frac{x^2}{2}}$, $x \in \mathbb{R}$.

- (2) Find the solution to the PDE

$$2x\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

of a function $u \equiv u(x, y)$ of two variables, such that $u(x, 0) = \sin(x)$, $x \in \mathbb{R}$.

- (3) Find the general solution to the PDE

$$(1 + x^2)\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} - u = e^y$$

of a function $u \equiv u(x, y)$ of two variables.

Exercise 2 (7pts)

Questions (1), (2), (3) of this exercise are independent.

- (1) What are the types of the following second-order linear PDE, where the unknown $u \equiv u(x, y)$ is a function of two variables?

(a) $\frac{\partial u}{\partial x} - 5\frac{\partial^2 u}{\partial x \partial y} + 2\frac{\partial u}{\partial y} + 12u = 0$.

(b) $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial y} + 2\frac{\partial^2 u}{\partial x \partial y} + 2\frac{\partial^2 u}{\partial y^2} = 0$.

(c) $\frac{\partial u}{\partial y} + 4\frac{\partial^2 u}{\partial x^2} - 4u = 0$.

- (2) Consider the following linear second-order PDE of a function $u \equiv u(x, y)$ of two variables:

$$x\frac{\partial^2 u}{\partial x^2} + (x + y)\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} = 0.$$

Find out the different regions in the space \mathbb{R}^2 of variables where the equation is elliptic, parabolic and hyperbolic. Sketch these regions.

- (3) Consider the following linear second-order PDE of a function $u \equiv u(x, y)$ of two variables:

(1)
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 3\frac{\partial u}{\partial x} - 4\frac{\partial u}{\partial y} + 25u = 0.$$

Our goal is to put this equation under the form:

(2)
$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + cv = 0,$$

for some constant c to be found by making a change of unknown function $u \mapsto v$. To this end, we search v under the form:

$$u(x, y) = e^{ax+by}v(x, y), \quad v(x, y) = e^{-ax-by}u(x, y),$$

where the constants $a, b \in \mathbb{R}$ are to be found.

- (a) Compute the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial y^2}$ in terms of those of v and a, b .

- (b) What values should we choose for a, b so that (1) is of the form (2)? What is then the value of c in (2)?

Exercise 3 (6 pts)

Consider a three dimensional rod oriented along the x -axis, of fixed cross-section area A . The rod is very thin, and lies in the region $(0 < x < L)$, so that we assume its temperature u only depends on the time t and on x . The specific heat per unit of mass c , density ρ and Fourier coefficient κ are assumed to be constant, and no sources are applied. The initial temperature distribution in the rod is described by a given function $\phi(x), x \in [0, L]$. The input and output heat flux are assumed to be known (Neumann boundary conditions):

$$\forall t > 0, \quad -\kappa \frac{\partial u}{\partial x}(t, 0) = \alpha, \quad -\kappa \frac{\partial u}{\partial x}(t, L) = \beta,$$

- (1) Write down the PDE satisfied by $u(t, x)$, together with the corresponding initial and boundary conditions. Explain if α and β correspond to a flux going in or out of the rod.
- (2) Express the total heat $H(t)$ in the rod at time t in terms of $A, c\rho, u, L$.
- (3) Show that $H'(t) = A(\alpha - \beta)$. Deduce the value of $H(t)$ at any time t in terms of A, α, β, t , and $H(0)$.
- (4) Does an equilibrium state always exist in this situation? If not, what is a condition for such a state to exist, and what is the physical meaning of such a condition?

Exercise 4 (7 pts)

Throughout this exercise, we consider the one-dimensional heat equation:

$$(3) \quad \frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = 0,$$

for $t > 0$ and $x \in (0, 1)$. This equation is supplemented by homogeneous Dirichlet boundary conditions:

$$(4) \quad \forall t > 0, \quad u(t, 0) = 0, \quad u(t, 1) = 0,$$

and the initial condition reads:

$$(5) \quad \forall x \in [0, 1], \quad u(0, x) = 4x(1 - x).$$

Let $u(t, x)$ be solution to this system.

- (1) State *precisely* the maximum principle, and the minimum principle for equation (3).
- (2) Show that, for any $t > 0$ and $0 \leq x \leq 1$, one has:

$$0 \leq u(t, x) \leq 1.$$

- (3) Show that, for any $t > 0$ and $0 \leq x \leq 1$, one has:

$$u(t, x) = u(t, 1 - x).$$

(Hint: show by a direct computation that $(t, x) \mapsto u(t, 1 - x)$ is also a solution to the system (3)-(4)-(5), then use the unicity of its solution)

- (4) Use the energy method to show that the energy $E(t) := \frac{1}{2} \int_0^1 u^2(t, x) dx$ is a decreasing function of time.

Exercise 5 (4 pts)

In this exercise, we consider the one-dimensional wave equation:

$$(6) \quad \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0,$$

for $t > 0, x \in \mathbb{R}$. This equation is equipped with the initial conditions:

$$(7) \quad \forall x \in \mathbb{R}, \quad u(0, x) = \phi(x), \quad \frac{\partial u}{\partial t}(0, x) = \psi(x),$$

where the functions ϕ, ψ are assumed to be given.

- (1) Recall the formula for the unique solution u to (6)-(7).
- (2) We thenceforth assume that $\phi(x) = \cos(x)$, $\psi(x) = 0$. Show that:

$$u(t, x) = \cos(x) \cos(ct).$$

- (3) Draw the solution $u(t, x)$ at the successive times $t = 0, \frac{\pi}{4c}, \frac{\pi}{2c}, \frac{3\pi}{4c}, \frac{\pi}{c}$. Report on each drawing as much information as possible (relevant values for x, u , etc...).