Exercise 1

We consider the heat equation $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$ on $t>0$, $x \in (0,\ell)$, together with boundary conditions

$\forall t>0$, $u(t,0) = 0$, $u(t,\ell) = e^{t}$, and initial conditions $u(0,x) = 0$, $x \in (0,\ell)$

To this end, we assume that u is twice differentiable, and expand the 3 functions u, $\frac{\partial u}{\partial t}$, $\frac{\partial^2 u}{\partial x^2}$ as sine Fourier series

$$u(t,x) = \sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{n\pi x}{\ell}\right).$$

$$\frac{\partial u}{\partial t}(t,x) = \sum_{n=1}^{\infty} v_n(t) \sin\left(\frac{n\pi x}{\ell}\right)$$

$$\frac{\partial^2 u}{\partial x^2}(t,x) = \sum_{n=1}^{\infty} w_n(t) \sin\left(\frac{n\pi x}{\ell}\right)$$

1) According to you, what legitimates the choice of expanding u, $\frac{\partial u}{\partial t}$, $\frac{\partial^2 u}{\partial x^2}$ as sine Fourier series?

We already know that the solution to the associated homogeneous equation, with homogeneous boundary conditions (of Dirichlet type) is naturally expressed as a sine Fourier expansion.
We can thus expect that sine Fourier series behave nicely when dealing with Dirichlet boundary conditions.

2) Express the coefficients $u_n(t)$, $v_n(t)$ and $w_n(t)$ in terms of $u(t,x)$

To this end, we simply use their definitions as the Fourier coefficients (in the sine expansion), associated to u, $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$ respectively:
The formulas for the coefficients yield:

$$\boxed{u_n(t) = \frac{2}{\ell}\int_0^{\ell} u(t,x) \sin\left(\frac{n\pi x}{\ell}\right) dx}$$

$$\boxed{v_n(t) = \frac{2}{\ell}\int_0^{\ell} \frac{\partial u}{\partial t}(t,x) \sin\left(\frac{n\pi x}{\ell}\right) dx}$$

$$\boxed{w_n(t) = \frac{2}{\ell}\int_0^{\ell} \frac{\partial^2 u}{\partial x^2}(t,x) \sin\left(\frac{n\pi x}{\ell}\right) dx.}$$

3) By using inversion of the integral sign and $\frac{\partial}{\partial t}$, or performing integration by parts, express $v_n(t)$ and $w_n(t)$ in terms of $u_n(t)$ (and the boundary condition).

For $v_n(t)$ this is just inverting the signs $\frac{\partial}{\partial t}$ and $\int_0^{\ell}$:

$$v_n(t) = \frac{2}{\ell}\int_0^{\ell} \frac{\partial u}{\partial t}(t,x) \sin\left(\frac{n\pi x}{\ell}\right) dx$$
$$= \frac{d}{dt}\left(\frac{2}{\ell}\int_0^{\ell} u(t,x) \sin\left(\frac{n\pi x}{\ell}\right) dx\right)$$
$$= \frac{d}{dt} u_n(t).$$

As far $w_n(t)$, we perform integration by parts:

$$w_n(t) = \frac{2}{\ell}\int_0^{\ell} \frac{\partial^2 u}{\partial x^2}(t,x) \sin\left(\frac{n\pi x}{\ell}\right) dx$$
$$= \frac{2}{\ell}\left(\cancel{\left[\frac{\partial u}{\partial x}(t,x) \sin\left(\frac{n\pi x}{\ell}\right)\right]_0^{\ell}} - \frac{n\pi}{\ell}\int_0^{\ell} \frac{\partial u}{\partial x}(t,x) \cos\left(\frac{n\pi x}{\ell}\right) dx\right)$$
$$= -\frac{2n\pi}{\ell^2}\int_0^{\ell} \frac{\partial u}{\partial x}(t,x) \cos\left(\frac{n\pi x}{\ell}\right) dx$$
$$= -\frac{2n\pi}{\ell^2}\left(\left[u(t,x)\cos\left(\frac{n\pi x}{\ell}\right)\right]_0^{\ell} + \frac{n\pi}{\ell}\int_0^{\ell} u(t,x) \sin\left(\frac{n\pi x}{\ell}\right) dx\right)$$
$$= -\frac{2n\pi}{\ell^2}\left((-1)^n e^{t} - 0\right) - \frac{2n^2\pi^2}{\ell^3}\int_0^{\ell} u(t,x)\sin\left(\frac{n\pi x}{\ell}\right) dx$$ by using the boundary conditions.
$$\boxed{= \frac{2n\pi}{\ell^2}(-1)^{n+1} e^{t} - \frac{n^2\pi^2}{\ell^2} u_n(t)}$$

4) Show that the fact that $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$ implies that for any $n \in \mathbb{N}^*$, one has: $\frac{d}{dt}u_n(t) + \frac{n^2\pi^2}{\ell^2}u_n(t) = \frac{2n\pi}{\ell^2}(-1)^{n+1}e^{t}$.

Using the expansions for $\frac{\partial u}{\partial t}$ & $\frac{\partial^2 u}{\partial x^2}$, we get:

$$\sum_{n=1}^{\infty} v_n(t) \sin\left(\frac{n\pi x}{\ell}\right) - \sum_{n=1}^{\infty} w_n(t) \sin\left(\frac{n\pi x}{\ell}\right) = 0.$$

i.e. $\sum_{n=1}^{\infty} (v_n(t) - w_n(t)) \sin\left(\frac{n\pi x}{\ell}\right) = 0$, and by unicity of the Fourier expansion of the function 0:

$\forall n \geq 1$: $v_n(t) - w_n(t) = 0$.

and, by the previous calculations: $\boxed{\frac{d}{dt}u_n(t) + \frac{n^2\pi^2}{\ell^2}u_n(t) = (-1)^{n+1}\frac{2n\pi}{\ell^2}e^{t}}$

5) For any real parameters $a, b \in \mathbb{R}$, solve the first-order ODE: $y'(t) + ay(t) = be^{t}$.

- We first solve the homogeneous equation; its solution is $y_{hom}(t) = Ce^{-at}$ for C real constant.
- To get the solution of the inhomogeneous equation, we use the method of variation of constants, and search for the solution $y(t)$ under the form: $y(t) = C(t)e^{-at}$, where the function $C(t)$ is to be found.

With this expression, we have: $y'(t) = C'(t)e^{-at} - aC(t)e^{-at}$
$= C'(t)e^{-at} - ay(t)$

and: $y'(t) + ay(t) = be^{t} \iff C'(t)e^{-at} = be^{t}$
i.e.: $C'(t) = be^{(1+a)t}$
$\Rightarrow C(t) = \frac{b}{1+a}e^{(1+a)t} + D$, for a constant $D \in \mathbb{R}$

and the general solution to the ODE is: $\boxed{y(t) = C(t)e^{-at} = \frac{b}{1+a}e^{t} + De^{-at},\ D \in \mathbb{R}.}$

6) Deduce from your calculations that $u_n(t) = \frac{(-1)^{n+1}2n\pi}{\ell^2 + n^2\pi^2}e^{t} + c_n e^{-\frac{n^2\pi^2 t}{\ell^2}}$, for some constant c_n to be found. Sketch the expansion for $u(t,x)$

This is just plugging $a = \frac{n^2\pi^2}{\ell^2}$, $b = (-1)^{n+1}\frac{2n\pi}{\ell^2}$ in the above expression.
The corresponding expansion for $u(t,x)$ is:

$$\boxed{u(t,x) = \sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}2n\pi}{\ell^2 + n^2\pi^2}e^{t} + c_n e^{-\frac{n^2\pi^2 t}{\ell^2}}\right)\sin\left(\frac{n\pi x}{\ell}\right)}$$

7) By using the initial condition $u(0,x) = 0$, $x \in (0,\ell)$, find the value of the c_n, and the expression of $u(t,x)$.

By putting $t=0$ in the previous formula:

$$u(0,x) = \sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}2n\pi}{\ell^2 + n^2\pi^2} + c_n\right)\sin\left(\frac{n\pi x}{\ell}\right) = 0.$$ and by unicity of the sine Fourier expansion of 0, we have:

$$c_n = -\frac{(-1)^{n+1}2n\pi}{\ell^2 + n^2\pi^2}$$

Eventually, we have: $\boxed{u(t,x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}2n\pi}{\ell^2 + n^2\pi^2}\left(e^{t} - e^{-\frac{n^2\pi^2 t}{\ell^2}}\right)\sin\left(\frac{n\pi x}{\ell}\right)}$

Let us recall Green's Formula: for any domain $\Omega \subseteq \mathbb{R}^3$, $\iiint_\Omega \operatorname{div}(V)\,dx = \iint_{\partial\Omega} V.n\,ds$, where $V:\Omega \to \mathbb{R}^3$ is a vector field : $V: \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}$.

. n is the normal vector field to $\partial\Omega$, pointing outward Ω.

. $\operatorname{div} V: \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$.

1) Recall that if $u:\Omega \to \mathbb{R}$ is a differentiable function, then $\nabla u: \Omega \to \mathbb{R}$ is defined as the vector field $\nabla u(x): \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \end{bmatrix}$

Show the formula; for any function $u:\Omega \to \mathbb{R}$
for any vector field $V: \Omega \to \mathbb{R}^3$

$\operatorname{div}(uV) : u \operatorname{div}(V) + \nabla u . V$.

This is a simple computation: $uV: \begin{bmatrix} uV_x \\ uV_y \\ uV_z \end{bmatrix}$, so: $\operatorname{div}(uV): \frac{\partial}{\partial x}(uV_x) + \frac{\partial}{\partial y}(uV_y) + \frac{\partial}{\partial z}(uV_z)$

$= \frac{\partial u}{\partial x} V_x + u\frac{\partial V_x}{\partial x} + \frac{\partial u}{\partial y} V_y + u\frac{\partial V_y}{\partial y} + \frac{\partial u}{\partial z} V_z + u\frac{\partial V_z}{\partial z}$

$= \left[\frac{\partial u}{\partial x}V_x + \frac{\partial u}{\partial y}V_y + \frac{\partial u}{\partial z}V_z\right] + u\left[\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right]$

$= \nabla u . V + u \operatorname{div}(V)$.

2) Show that for any two functions $u,v:\Omega \to \mathbb{R}$, $\iiint_\Omega u \Delta v\,dx = \iint_{\partial\Omega} u \frac{\partial v}{\partial n} ds - \iiint_\Omega \nabla u . \nabla v\,dx$

We apply Green's formula to the vector field $u\nabla v$.

We have: $\iiint_\Omega \operatorname{div}(u\nabla v)\,dx : \iint_{\partial\Omega} u . \nabla v . n\,ds = \iint_{\partial\Omega} u \frac{\partial v}{\partial n} ds$. and, using Question 1): $\operatorname{div}(u\nabla v) : \nabla u . \nabla v + u \operatorname{div}(\nabla v)$
$: \nabla u . \nabla v + u\Delta v$.

Thus, $\iiint_\Omega \nabla u . \nabla v\,dx + \iiint_\Omega u\Delta v\,dx : \iint_{\partial\Omega} u \frac{\partial v}{\partial n} ds$; and the result follows.

3) We now consider the Laplace equation $\Delta u : f$ in Ω, with inhomogeneous boundary conditions $u : g$ on $\partial\Omega$.

Show unicity of the solution to this system.

Assume we have two solutions u, v to this system, and take their difference $w := u - v$.

Then w satisfies: $\begin{cases} \Delta w : 0 \text{ in } \Omega. \\ w : 0 \text{ on } \partial\Omega \end{cases}$

multiplying $\Delta w : 0$ by w and integrating yields: $\iiint_\Omega w \Delta w\,dx : 0$, and using the previous formula:

$\underbrace{\iint_{\partial\Omega} w \frac{\partial w}{\partial n} ds}_{= 0 \text{ because } w : 0 \text{ on } \partial\Omega} - \iiint_\Omega \nabla w . \nabla w\,dx : 0$.

Thus: $\iiint_\Omega \|\nabla w\|^2 dx = 0$ The function $\|\nabla w\|^2$ is continuous, positive, with 0 integral over Ω $\Rightarrow$ $\nabla w = 0$ on Ω.
The gradient of w being 0 on Ω, w is constant on Ω: $w : c$, where $c \in \mathbb{R}$.
But $w : 0$ on $\partial\Omega$, so $c : 0$.
$\boxed{\text{finally } w : 0}$.

4) We consider the same equation with non-homogeneous Neumann BC: $\frac{\partial u}{\partial n} : g$ on $\partial\Omega$. Show that there is unicity of solutions to the new system, up to constants.

Let u, v be two solutions, and $w := u - v$. Our goal is to prove that w : constant. (w satisfies: $\begin{cases} \Delta w : 0 \text{ on } \Omega. \\ \frac{\partial w}{\partial n} : 0 \text{ on } \partial\Omega \end{cases}$)

To this end, once again, we multiply $-\Delta w : 0$ by w, and integrate:

$\iiint_\Omega w \Delta w = 0$ so: $\underbrace{\iint_{\partial\Omega} w \frac{\partial w}{\partial n} ds}_{= 0 \text{ on } \partial\Omega} - \iiint_\Omega \|\nabla w\|^2 dx : 0$. , so that, again, $\iiint_\Omega \|\nabla w\|^2 dx : 0$, and for the same reasons as before: $\nabla w : 0 \Rightarrow w$: constant on Ω.

5) We eventually assume that u satisfies inhomogeneous Robin BC: $\frac{\partial u}{\partial n} + \alpha u : g$, where $\alpha > 0$. Show unicity of solutions.

Again, let u, v two solutions, and $w : u - v$.

Then: $\begin{cases} \Delta w : 0 \text{ on } \Omega \\ \frac{\partial w}{\partial n} - \alpha w : 0 \text{ on } \partial\Omega \end{cases}$

Using the same trick as before: $\iiint_\Omega w \Delta w\,dx : 0 \Rightarrow \iint_{\partial\Omega} w \frac{\partial w}{\partial n} ds - \iiint_\Omega \|\nabla w\|^2 dx : 0$.

so: $-\alpha \iint_{\partial\Omega} w^2 dx - \iiint_\Omega \|\nabla w\|^2 dx \leq 0$. Thus $\iiint_\Omega \|\nabla w\|^2 dx : -\alpha \iint_{\partial\Omega} w^2 dx \leq 0$.

We conclude that, actually $\iiint_\Omega \|\nabla w\|^2 dx : 0$. , so that, also $\iint_{\partial\Omega} w^2 ds : 0$.
So $\nabla w : 0$ on Ω, and w is a constant.
But, also, $\iint_{\partial\Omega} w^2 ds : 0 \Rightarrow w : 0$ on $\partial\Omega$, and the constant is 0.

$\boxed{\text{Exercise 3}}$ SW 6.1.9.

In two dimensions, we consider the domain $\Omega : \{ x \in \mathbb{R}^2, 1 \leq \|x\| \leq 2 \}$, i.e. the annulus of inner radius 1, and of outer radius 2.

Remember that the temperature inside the annulus satisfies the heat equation $\frac{\partial u}{\partial t} - k\Delta u : 0$ in Ω.

In this exercise, we are interested in the steady for the temperature, which satisfies $\Delta u : 0$.

The boundary conditions are : $u : 100$ on the inner circle,
$\frac{\partial u}{\partial n} : -\gamma$, $\gamma > 0$ in the outer radius, and we search for a radially symmetric solution $u : u(x,y) : u(r)$.

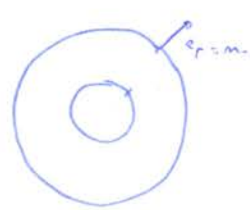
1) According to you why searching for a radially symmetric solution?

We know that the Laplace operator is invariant by rotation. Because the BC and sources (0 here) are also invariant to rotation, we can expect the solution to be so.

2) Interpret the sign of γ in terms of energy flow.

Here, we have $\frac{\partial u}{\partial r} : + \frac{\partial u}{\partial n}$ on the outer radius.

Thus $\underbrace{-\frac{\partial u}{\partial n}}_{= \text{ heat flux by Fourier's law.}} : +\gamma$ the energy $+\gamma$ is lost by the system, so that $\gamma > 0$ is lost by the annulus.



3) Recall the form of the Laplace operator in polar coordinates: $\Delta u : \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$.

Solve the equation for u.

Because of $u = u(r)$, we have $\Delta u : 0 \Rightarrow \frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) : 0$. so that $\frac{d}{dr}\left(r\frac{du}{dr}\right) : 0$.

Hence $r\frac{du}{dr} : a$, where a is a constant

$\frac{du}{dr} : \frac{a}{r}$. and $u(r) : a\ln r + b$, where a and b are two constants.

eqs $u(1) : 100 \Rightarrow a\ln 1 + b : b : 100$ and $\left.\frac{du}{dr}\right|_{r:2} = \frac{a}{2} : -\gamma$, so that $a : -2\gamma$

Hence $\boxed{u(r) = -2\gamma \ln r + 100}$

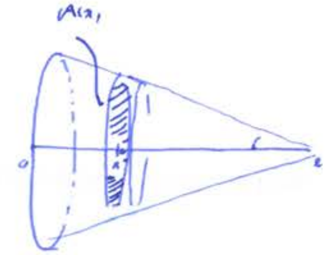
4) What are the hottest and coldest temperatures? Is this in agreement with the maximum and minimum principles?

By the look of the function, the hottest temperature is reached at the inner boundary and is worth $100 + 2\gamma \ln 2$.
The coldest temperature is reached on the outer boundary and is worth 100.
Both minimum and maximum values of u are reached on the boundary of Ω (and only on this boundary), which is exactly what is stated by the max / min principles.

5) Is there a way to choose γ so that the temperature at the outer boundary is 20°?

We must choose γ so that $100 - 2\gamma \ln 2 = 20$;
whence $\gamma = \dfrac{100-20}{2\ln 2} = \boxed{\dfrac{40}{\ln 2}}$; yes it is possible!

Exercise 4 | St. 5.6.6



Consider a 3d-rod, oriented along the x-axis, lying between $0 < x < \ell$.
The cross-sectional area of the rod at point x is $A(x) = b(1 - \frac{x}{\ell})^2$, where b is a fixed constant.
The rod is insulated at its lateral sides, kept at temperature 0 at both ends.
It is homogeneous, with coefficients $c, \rho = k = 1$.
We assume that the temperature is only a function of x.

1) Consider the slice lying between x and $x+h$. Calculate its internal energy, and express the variation of energy in the slice.

Its internal energy is $H(t) = \int_x^{x+h} A(r)\, u(t,r)\, dr$, (because $c = 1$, $\rho = 1$)

The instantaneous variation of energy of this slice is given by: $\begin{cases} \text{the addition of } -kA(x)\frac{\partial u}{\partial x}\Big|_x = -A(x)\frac{\partial u}{\partial x}\Big|_x, \text{ which is gained by the system.} \\ \text{the departure of the energy } -kA(x+h)\frac{\partial u}{\partial x}\Big|_{x+h} = -A(x+h)\frac{\partial u}{\partial x}\Big|_{x+h}, \text{ which leaves the system.} \end{cases}$

Hence: $\frac{d}{dt}H(t) = A(x+h)\frac{\partial u}{\partial x}\Big|_{x+h} - A(x)\frac{\partial u}{\partial x}\Big|_x$.

i.e.: $\int_x^{x+h} A(r)\frac{\partial u}{\partial t}(t,r)\,dr = A(x+h)\frac{\partial u}{\partial x}\Big|_{x+h} - A(x)\frac{\partial u}{\partial x}\Big|_x$.

Now taking the derivative with respect to h and evaluating at $h=0$ leads to: $\boxed{A(x)\frac{\partial u}{\partial t}(t,x) = \frac{\partial}{\partial x}\left(A(x)\frac{\partial u}{\partial x}\right)(t,x)}$

2) We search to solve this equation by the method of separation of variables, and first search for the separated solutions to the system: $u(t,x) = T(t)X(x)$.
Write down the equation (and BC) satisfied by T and X.

We have, as usual: $A(x)T'(t)X(x) = \frac{d}{dx}(A(x)X'(x))T(t)$

Thus: $-\dfrac{T'(t)}{T(t)} = -\dfrac{\frac{d}{dx}(A(x)X'(x))}{A(x)X(x)}$ for all $t>0$, $x \in (0,\ell)$.

As this figures an equality between a function of t only and a function of x only, which is valid for $t>0$, $x \in (0,\ell)$, then both are actually constants: there exists $\lambda \in \mathbb{R}$ such that

$-\dfrac{T'(t)}{T(t)} = -\dfrac{\frac{d}{dx}(A(x)X'(x))}{A(x)X(x)} = \lambda$, whence: $\begin{cases} T'(t) + \lambda T(t) = 0 \\ \frac{d}{dx}(A(x)X'(x)) + \lambda A(x)X(x) = 0 \end{cases}$, and the BC are $X(0) = X(\ell) = 0$.

3) By using the change of unknown function $v(x) = (1-\frac{x}{\ell})X(x)$, rewrite the ODE over X in terms of v.

We have to compute $X'(x)$ in terms of v:

$X(x) = \dfrac{v(x)}{1-\frac{x}{\ell}}$, thus: $X'(x) = \dfrac{v'(x)(1-\frac{x}{\ell}) + \frac{v(x)}{\ell}}{(1-\frac{x}{\ell})^2}$, and $A(x)X'(x) = b(1-\frac{x}{\ell})^2 X'(x) = b\left(v'(x)(1-\frac{x}{\ell}) + \frac{v(x)}{\ell}\right)$.

Now: $\frac{d}{dx}(A(x)X'(x)) = b\frac{d}{dx}\left(v'(x)(1-\frac{x}{\ell}) + \frac{v(x)}{\ell}\right) = b\left(v''(x)(1-\frac{x}{\ell}) - \cancel{\frac{1}{\ell}v'(x)} + \cancel{\frac{v'(x)}{\ell}}\right) = b(1-\frac{x}{\ell})v''(x)$.

On the other hand: $A(x)X(x) = b(1-\frac{x}{\ell})v(x)$.

Finally: $\frac{d}{dx}(A(x)X'(x)) + \lambda A(x)X(x) = 0$ becomes $b(1-\frac{x}{\ell})v''(x) + \lambda b(1-\frac{x}{\ell})v(x) = 0$

$\boxed{\text{i.e.: } v''(x) + \lambda v(x) = 0}$ and, we have the BC: $v(0) = b(1-0)X(0) = 0$.
$v(\ell) = b(1-1)X(\ell) = 0$.

4) Search for the negative eigenvalues of the problem. $\lambda = -\beta^2$.

This is the same thing as we did many times; there is no negative eigenvalue.

5) Is 0 an eigenvalue of the problem?

As in the lectures, no.

6) Search for the positive eigenvalues of the problem $\lambda = \beta^2$, as well as for the corresponding eigenfunctions $v_n(x)$, and $X_n(x)$.

As in the lectures (this is the eigenvalue problem associated with homogeneous Dirichlet BC), we find that $\beta_n = \frac{n\pi}{\ell}$, so that $\lambda_n = (\frac{n\pi}{\ell})^2$. The corresponding eigenfunctions are $v_n(x) = \sin(\frac{n\pi x}{\ell})$.

so that $\boxed{X_n(x) = \dfrac{\sin(\frac{n\pi x}{\ell})}{1-\frac{x}{\ell}}}$.

7) Solve for the corresponding temporal part T_n associated to the eigenvalue λ_n, and write down the expansion for the most general form of solutions to the system you have found.

T_n is solution to the ODE: $T_n'(t) + \lambda_n T_n(t) = 0$

so that $T_n(t) = b_n e^{-\frac{n^2\pi^2}{\ell^2}t}$, for some constant b_n to be determined.

Hence, the most general solution we have found for the considered system, without taking into account the initial conditions is:

$\boxed{u(t,x) = \sum_{m=1}^{\infty} b_m \frac{\sin(\frac{m\pi x}{\ell})}{1-\frac{x}{\ell}} e^{-\frac{m^2\pi^2}{\ell^2}t}, \text{ for constants } b_m.}$

8) We now come to identifying the coefficients b_m, by using the initial condition. Show that $b_m = \frac{2}{\ell}\int_0^\ell \phi(x)(1-\frac{x}{\ell})\sin(\frac{m\pi x}{\ell})\,dx$.

The initial conditions read: $\sum_{m=1}^{\infty} b_m \dfrac{\sin(\frac{m\pi x}{\ell})}{1-\frac{x}{\ell}} = \phi(x)$, for $x \in (0,\ell)$. We want to compute the coefficient b_m, for some given $m \in \mathbb{N}^*$.

The idea is to multiply the previous relation by $(1-\frac{x}{\ell})\sin(\frac{m\pi x}{\ell})$, integrating, and then using the usual orthogonality relation $\int_0^\ell \sin(\frac{n\pi x}{\ell})\sin(\frac{m\pi x}{\ell})\,dx = \begin{cases} 0 \text{ if } n \neq m \\ \frac{\ell}{2} \text{ if } n = m \end{cases}$

We have: $\sum_{n=1}^{\infty} b_n \dfrac{\sin(\frac{n\pi x}{\ell})}{\cancel{1-\frac{x}{\ell}}} \cdot \cancel{(1-\frac{x}{\ell})}\sin(\frac{m\pi x}{\ell}) = \phi(x)(1-\frac{x}{\ell})\sin(\frac{m\pi x}{\ell})$.

$\Rightarrow \sum_{n=1}^{\infty} b_n \int_0^\ell \sin(\frac{n\pi x}{\ell})\sin(\frac{m\pi x}{\ell})\,dx = \int_0^\ell \phi(x)(1-\frac{x}{\ell})\sin(\frac{m\pi x}{\ell})\,dx$, where we have used the interchanging between the infinite sum and the $\int$ signs.

Thus: $b_m \cdot \frac{\ell}{2} = \int_0^{\ell} \phi(x)\left(1-\frac{x}{\ell}\right)\sin\left(\frac{m\pi x}{\ell}\right)dx$. , and the desired result follows.

$\boxed{\text{Exercise 5}}$

The purpose of this exercise is to solve the inhomogeneous equation $\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = e^t \sin 5x$, for $0 < x < \pi$,

together with boundary conditions $u(t,0) = u(t,\pi) = 0$

and initial conditions $u(0,x) = 0$, $\frac{\partial u}{\partial t}(0,x) = \sin 3x$.

1) To go back into the framework of homogeneous PDE, we search to substract some adequate function to u.

We search for a change of variables function of the form $v(t,x) = u(t,x) + a e^t \sin 5x$, for some constant $a \in \mathbb{R}$.

Find the value of a such that v solves the homogeneous wave equation.

We compute the derivatives of u in terms of those of v: $\frac{\partial^2 u}{\partial t^2}(t,x) = \frac{\partial^2 v}{\partial t^2} - a e^t \sin(5x)$, since $u(t,x) = v(t,x) - a e^t \sin(5x)$.

and $\frac{\partial^2 u}{\partial x^2}(t,x) = \frac{\partial^2 v}{\partial x^2}(t,x) + 25 a e^t \sin(5x)$

Thus: $\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = e^t \sin(5x) \Rightarrow \frac{\partial^2 v}{\partial t^2} - c^2\frac{\partial^2 v}{\partial x^2} - a e^t \sin(5x) - 25ac^2 e^t \sin(5x) = e^t \sin(5x)$

i.e. $\frac{\partial^2 v}{\partial t^2} - c^2\frac{\partial^2 v}{\partial x^2} = (a + 25ac^2 + 1)e^t \sin(5x)$. Thus, taking a such that $a + 25ac^2 + 1 = 0$ (i.e.: $a = \frac{-1}{1+25c^2}$) leads to: $\frac{\partial^2 v}{\partial t^2} - c^2\frac{\partial^2 v}{\partial x^2} = 0$.

Plus, v satisfies the BC: $v(t,0) = u(t,0) - ae^t \sin(0) = 0$

$v(t,\pi) = u(t,\pi) - ae^t \sin(5\pi) = 0$.

and the IC: $v(0,x) = u(0,x) - a\sin(5x) = -a\sin(5x)$

$\frac{\partial v}{\partial t}(0,x) = \frac{\partial u}{\partial t}(0,x) - a\sin(5x) = \sin(3x) - a\sin(5x)$

2) Write down the general form solution to the system satisfied by v, without taking into account the IC, as a Fourier expansion

(as derived by the method of sep of var. you may use the result of the lecture instead of repeating the argument)

We have: $v(t,x) = \sum_{n=1}^{\infty}\left(a_n\cos(nct) + b_n\sin(nct)\right)\sin(nx)$

where the constants are to be found

3) By using the first initial condition for v, calculate the coefficients a_n.

We have: $v(0,x) = a\sin(5x)$; which implies: $\sum_{n=1}^{\infty} a_n \sin(nx) = a\sin(5x)$.

By unicity of the Fourier decomposition of the function $a\sin(5x)$, we have that all the $a_n = 0$, except for a_5, which equals a.

4) To use the second initial condition, recall that you mustn't derivate under the sign sum!

Write down the sine Fourier expansion of $\frac{\partial v}{\partial t}(t,x)$ as $\sum_{n=1}^{\infty} v_n(t)\sin(nx)$, and calculate $v_n(t)$ in terms of the b_n. Evaluate $v_n(0)$

By definition, the Fourier coefficient $v_n(t)$ is: $v_n(t) = \frac{2}{\pi}\int_0^{\pi}\frac{\partial v}{\partial t}(t,x)\sin(nx)\,dx$

$= \frac{d}{dt}\left(\frac{2}{\pi}\int_0^{\pi} v(t,x)\sin(nx)\,dx\right)$; but we know that $\frac{2}{\pi}\int_0^{\pi} v(t,x)\sin(nx)\,dx = a_n\cos(nct) + b_n\sin(nct)$,

since it is the sine Fourier coefficient of $v(t,x)$ (for a given t).

Thus: $v_n(t) = \frac{d}{dt}\left(a_n\cos(n c t) + b_n\sin(n c t)\right)$

$= n c\left(-a_n\sin(n c t) + b_n\cos(n c t)\right)$

In particular $\boxed{v_n(0) = n c\, b_n.}$

5) By using the second IC with this expansion (remember that you mustn't derivate under the Σ!). compute the b_n.

We have by IC: $\sum_{n=1}^{\infty} v_n(0)\sin(nx) = \sin(3x) + a\sin(5x) = \frac{\partial v}{\partial t}(0,x)$

i.e., thanks to the last question: $\sum_{n=1}^{\infty} n c\, b_n \sin(nx) = \sin(3x) + a\sin(5x)$

Thus, as before, all the $b_n = 0$ except $\begin{cases} b_3 = \frac{1}{3c} \\ b_5 = \frac{a}{5c}. \end{cases}$

6) Conclude as for the expression of $v(t,x)$, and that of $u(t,x)$

We have $v(t,x) = \sum_{n=1}^{\infty}\left(a_n\cos(nct) + b_n\sin(nct)\right)\sin(nx)$, but only a_5, b_3 and b_5 are $\neq 0$ in this expansion.

$\boxed{\text{Thus } v(t,x) = \frac{1}{3c}\sin(3ct)\sin(3x) + \frac{a}{5c}\sin(5ct)\sin(5x) + a\cos(5ct)\sin(5x), \text{ with } a = \frac{-1}{1+25c^2}}$

$\boxed{\text{and } u(t,x) = v(t,x) - ae^t\sin(5x),}$ by definition.