Elementary partial differential equations: homework 8

Assigned 04/08/2014, due 04/22/2014.

Exercise 1

This exercise is reprinted from [Strauss], §5.4, Exercise 5.
Let $\phi : [0,3] \to \mathbb{R}$ be the function defined by:

$$\forall x \in [0,3],\ \ \phi(x) = \begin{cases} 0 & \text{for } 0 < x < 1 \\ 1 & \text{for } 1 < x < 3 \end{cases}$$

(1) Calculate explicitly the first 4 nonzero coefficients appearing in the Fourier cosine series expansion

$$\phi(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{3})$$

of ϕ over $[0,3]$.

(2) By using the theorem of the lectures about pointwise convergence of Fourier series, calculate explicitly the pointwise limit of the series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{3})$ for any $0 \leq x \leq 3$.

[Warning: Remember that the study of the pointwise convergence of Fourier series only applies to full Fourier series.]

(3) Does this cosine series converge in the L^2 sense?

(4) By using the pointwise convergence result of question (2) at $x = 0$, compute the infinite sum

$$1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \frac{1}{10} - \frac{1}{11} + \ldots$$

Exercise 2

Let us consider the 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$, defined over $[0, 2\pi[$ as:

$$\forall x \in [0, 2\pi[,\ \ f(x) = \begin{cases} \pi - x & \text{for } 0 < x < \pi \\ 0 & \text{for } \pi < x < 2\pi \end{cases}$$

(1) Draw the graph of f over several periods.

(2) Show that the full Fourier series of f writes:

$$f(x) = \frac{\pi}{4} + \sum_{p=0}^{\infty} \frac{2}{\pi(2p+1)^2} \cos((2p+1)x) + \sum_{n=1}^{\infty} \frac{\sin(nx)}{n}.$$

(3) Compute the pointwise limit of this full Fourier series for any $x \in [-\pi, \pi]$.

(4) By using the pointwise convergence result of the previous question for $x = 0$, calculate the series $\sum_{p=0}^{\infty} \frac{1}{(2p+1)^2}$.

(5) Calculate the value of $\sin(\frac{n\pi}{2})$, depending on the parity of n, and use this result to compute the series $\sum_{p=0}^{\infty} \frac{(-1)^p}{2p+1}$ by using the pointwise convergence result of Question (3) at a particular point.

(6) Show that, for any $x \in]0, \pi]$, the following equality holds:

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n} = \frac{\pi}{4} + \frac{2}{\pi} \sum_{p=0}^{\infty} \frac{\cos((2p+1)(x))}{(2p+1)^2}.$$

Exercise 3

(1) Let $\ell > 0$; compute the Fourier sine series of the function $\phi(x) = x$ over the interval $[0, \ell]$.

(2) Does this series converge in the L^2 sense?

(3) Apply Parseval's formula to compute the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Exercise 4

Consider the function $\phi : [-\pi, \pi] \to \mathbb{R}$ defined as: $\phi(x) = |x|$. We want to approximate ϕ over $[-\pi, \pi]$ by a function

$$f(x) = \frac{a_0}{2} + a_1 \cos(x) + b_1 \sin(x) + a_2 \cos(2x) + b_2 \sin(2x),$$

where the coefficients $a_0, a_1, b_1, a_2, b_2 \in \mathbb{R}$ are to be found, so as to minimize the L^2 discrepancy

$$||\phi - f||_{L^2} = \sqrt{\int_{-\pi}^{\pi} (\phi - f)^2)\, dx}.$$

By using the contents of the lecture over the L^2 theory for Fourier series, show that the best choice as for these constants is:

$$a_0 = \pi,\;\; a_1 = -\frac{4}{\pi},\;\; b_1 = a_2 = b_2 = 0.$$

Exercise 5

This exercise is reprinted from [Strauss], §5.4, Exercise 8.

Let $\ell > 0$; for each of the following functions f, consider its sine Fourier series over the interval $(0, \ell)$ (it is neither asked, nor necessary to compute this series). Indicate whether this series converge uniformly, pointwise (and in this case, calculate the pointwise limit at any point of the interval, including the endpoints), and in the L^2 sense:

(1) $f(x) = x^3$,
(2) $f(x) = \ell x - x^2$,
(3) $f(x) = \frac{1}{x^2}$.

Exercise 6

(1) Let $\ell > 0$; compute the cosine Fourier series of $f(x) = x^2$ over $(0, \ell)$.
(2) By using Parseval's formula, calculate the sum $\frac{1}{n^4}$.
[Warning: Remember to be careful about how to deal with the term for $n = 0$ with Parseval's formula].