Elementary partial differential equations: homework 7

Assigned 04/01/2014, due 04/08/2014.

Exercise 1

This exercise is reprinted from [Strauss], §5.2, Exercise 11.

Let $\ell > 0$ be a positive real number, and $\phi : [-\ell, \ell] \to \mathbb{R}$ be the function defined by:

$$\forall x \in [-\ell, \ell], \ \ \phi(x) = e^x.$$

(1) Calculate the coefficients of the full Fourier series of ϕ over the interval $[-\ell, \ell]$ under their complex form, i.e. those coefficients $c_n \in \mathbb{C}$, $n = -\infty, ..., 0, ..., \infty$ appearing in the series expansion:

$$\phi(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{\ell}}.$$

(2) Deduce from the result of Question (1) the expression of the coefficients of the full Fourier series of ϕ over $[-\ell, \ell]$ under their real form, i.e. those coefficients $a_n \in \mathbb{R}$, $n = 0, .., \infty$, and $b_n \in \mathbb{R}$, $n = 1, ..., \infty$ appearing in the expansion:

$$\phi(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right).$$

Exercise 2

This exercise is reprinted from [Strauss], §5.2, Exercise 1.

For each of the following functions$f : \mathbb{R} \to \mathbb{R}$, indicate whether it is even, odd, or periodic. In this last case, specify also the smallest period of the considered function.

(1) $f(x) = \sin(ax)$, where $a > 0$ is fixed.
(2) $f(x) = e^{ax}$, where $a > 0$ is fixed.
(3) $f(x) = x^m$, where $m \in \mathbb{N}$ is fixed.
(4) $f(x) = \tan(x^2)$.
(5) $f(x) = |\sin(\frac{x}{a})|$, where $a > 0$ is fixed.
(6) $f(x) = x\cos(ax)$, where $a > 0$ is fixed.

Exercise 3

This exercise is reprinted from [Strauss], §5.2, Exercise 10.

Throughout this exercise, $\ell > 0$ stands for a positive real number, and $\phi : [0, \ell] \to \mathbb{R}$ is a *continuous* function.

(1) Let $\phi_{\text{odd}} : [-\ell, \ell] \to \mathbb{R}$ be the odd extension of ϕ. Under what condition(s) on ϕ is ϕ_{odd} a continuous function over $[-\ell, \ell]$?
(2) Let $\phi_{\text{even}} : [-\ell, \ell] \to \mathbb{R}$ be the even extension of ϕ. Under what condition(s) on ϕ is ϕ_{even} a continuous function over $[-\ell, \ell]$?
(3) Under the additional assumption that ϕ is differentiable over $[0, \ell]$, under what condition(s) on ϕ is the odd extension ϕ_{odd} a differentiable function over $[-\ell, \ell]$?
(4) Still under the assumption that ϕ is differentiable over $[0, \ell]$, under what condition(s) on ϕ is the even extension ϕ_{even} a differentiable function over $[-\ell, \ell]$?

Exercise 4

This exercise is reprinted from [Strauss], §5.2, Exercise 15.

Without performing any computation, predict which of the coefficients in the full Fourier series expansion of the function

$$\forall x \in [-\pi, \pi], \;\; \phi(x) = |\sin x|$$

must vanish.

Exercise 5

This exercise is partly reprinted from [Strauss], §5.1, Exercise 5.

Let $\ell > 0$ be a positive real number, and consider the function $\phi(x) = x$ over the interval $[0, \ell]$.

(1) Calculate the coefficients of the sine Fourier series of ϕ over $[0, \ell]$ and write down the corresponding sine Fourier series expansion.

(2) We now assume that the sine Fourier series expansion $\phi(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right)$ of ϕ can be integrated term by term, that is, for any points $a, b \in [0, \ell]$, one has:

$$\int_a^b \phi(x)\, dx = \int_a^b \left(\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right) \right) dx = \sum_{n=1}^{\infty} b_n \left(\int_a^b \sin\left(\frac{n\pi x}{\ell}\right) dx \right),$$

a result which shall be proven in the forthcoming lectures. Infer from the result of Question (1) the cosine Fourier series expansion of the function $x \mapsto \frac{x^2}{2}$.

(3) By calculating the constant coefficient a_0 in the obtained expansion in Question (2) by two different means, show that:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

Exercise 6

This exercise is partly reprinted from [Strauss], §5.3, Exercise 6.

(1) Find all the complex values $\lambda \in \mathbb{C}$ such that $e^{\lambda} = 1$.

(2) Consider the operator $X \mapsto X'$, defined over functions $X : [0, 1] \to \mathbb{R}$ of one single (real) variable such that $X(0) = X(1)$. Using Question (1), find all its complex eigenvalues λ_n, $n = 1, ..., \infty$, as well as the corresponding eigenfunctions X_n.

(3) Show by a direct calculation that two eigenfunctions X_n, X_m, associated to two *different* eigenvalues $\lambda_n \neq \lambda_m$ are orthogonal, i.e.:

$$\int_0^1 X_n(x)\overline{X_m(x)}\, dx = 0.$$

Exercise 7

This exercise is partly reprinted from [Strauss], §5.1, Exercise 8.

Consider a one-dimensional rod of length $\ell = 1$, and heat coefficient $\kappa = 1$. We assume that the temperature $u(t, x)$ at time $t > 0$ and position $x \in [0, 1]$ obeys the heat equation:

$$\forall t > 0, \;\; \forall x \in (0, 1), \;\; \frac{\partial u}{\partial t}(t, x) - \frac{\partial^2 u}{\partial x^2}(t, x) = 0.$$

The length end of the rod is kept at constant temperature 0 and the right end at temperature 1, so that $u(t, x)$ enjoys non homogeneous Dirichlet boundary conditions:

$$\forall t > 0, \;\; u(t, 0) = 0, \;\; u(t, 1) = 1.$$

The initial temperature distribution in the rod is given by:

$$\forall x \in [0, 1], \;\; u(0, x) = \phi(x) := \begin{cases} \frac{5x}{2} & \text{for } 0 < x < \frac{2}{3} \\ 3 - 2x & \text{for } \frac{2}{3} < x < 1 \end{cases}.$$

(1) Find the equilibrium temperature distribution $U : [0,1] \to \mathbb{R}$ in the rod.

(2) Consider the function $v(t,x) = u(t,x) - U(x)$. Show that v satisfies the heat equation $\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = 0$, together with *homogeneous* Dirichlet boundary conditions:

$$\forall t > 0, \;\; v(t,0) = 0, \;\; v(t,1) = 0,$$

and with the initial condition:

$$\forall x \in [0,1], \;\; v(0,x) = \psi(x) := \begin{cases} \frac{3x}{2} & \text{for } 0 < x < \frac{2}{3} \\ 3(1-x) & \text{for } \frac{2}{3} < x < 1 \end{cases} .$$

(3) Calculate the sine Fourier expansion of ψ over $[0,1]$.

(4) By using the method of separation of variables exactly as in the lectures, find the expression of $v(t,x)$ as a series expansion.

(5) From the result of the previous question, find the expression of $u(t,x)$ as a series expansion.