Elementary partial differential equations: homework 6

Assigned 03/25/2014, due 04/01/2014.

Exercise 1

This exercise is partially reprinted from [Strauss], §5.1, Exercise 2.

Let $\phi : [0,1] \to \mathbb{R}$ be the function defined by:

$$\forall x \in [0,1], \;\; \phi(x) = x^2.$$

(1) Calculate the Fourier sine series of ϕ over the interval $[0,1]$.
(2) Calculate is Fourier cosine series over the interval $[0,1]$.
(3) Remember from the lectures that the Fourier sine or cosine series of a function do not necessarily converge to this function. We now *assume* that the Fourier cosine series of ϕ converges towards ϕ on the whole interval $[0,1]$. Use the answer to question (2) to show that:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}.$$

Exercise 2

This exercise is reprinted from [Strauss], §5.1, Exercise 4.

(1) Compute the full Fourier series of the function $\phi : [-\pi,\pi] \to \mathbb{R}$ defined by:

$$\forall x \in [-\pi,\pi], \;\; \phi(x) = |\sin x|.$$

(2) We now *assume* that the full Fourier series of ϕ converges towards ϕ on $[-\pi,\pi]$. Use the result of Question (1) to compute the values of the sums:

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}.$$

Exercise 3

This exercise is partially reprinted from [Strauss], §4.3, Exercise 9.

In this exercise, we consider the heat equation on the interval $[0,1]$:

$$\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = 0, \tag{1}$$

with boundary conditions:

$$\forall t > 0, \;\; \frac{\partial u}{\partial x}(t,0) + u(t,0) = 0, \quad \text{and } u(t,1) = 0. \tag{2}$$

(1) Use the method of separation of variables and write down the two ODE satisfied by the temporal and spatial parts of a separated solution $u(t,x) = T(t)X(x)$, as in the lectures. In particular, show that there exists $\lambda \in \mathbb{R}$ such that the spatial part $X : [0,1] \to \mathbb{R}$ of a separated solution should satisfy:

$$\begin{cases} -X''(x) = \lambda X(x) & \text{for } x \in (0,1) \\ X'(0) + X(0) = 0, & X(1) = 0 \end{cases}. \tag{3}$$

(2) Find an eigenfunction associated to the eigenvalue $\lambda = 0$. Call it $X_0(x)$.
(3) Find an equation for the positive eigenvalues $\lambda = \beta^2$, $\beta > 0$.
(4) Show graphically, as during the lectures, that there are an infinite number of positive eigenvalues. What are the associated eigenfunctions?

(5) Does the system (3) have a negative eigenvalue?
(6) Solve the temporal equation associated to each eigenvalue λ and write down the general series expansion for solutions to the system (1 - 2).

Exercise 4

This exercise is partially reprinted from [Haberman], §2.3, Exercise 3.

Consider a three dimensional rod oriented along the x-axis, with cross-sectional area A and lateral perimeter P (see Figure 1).

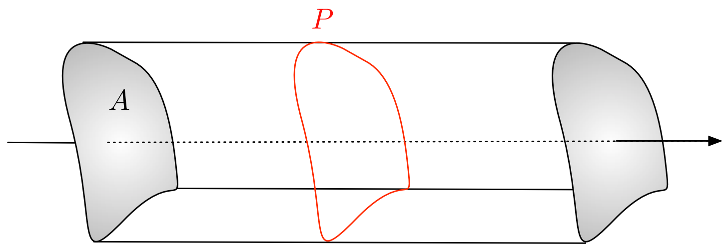

FIGURE 1. Setting for Exercise 4.

The rod is very thin, and lies in the region $(0 < x < \ell)$, so that we assume that its temperature u only depends on the time t and on x; it is insulated at both ends (homogeneous Dirichlet boundary conditions). Its (constant) density is denoted as ρ, its (constant) specific heat as c, and its (constant) Fourier constant as κ.

The rod does not contain any thermal source, but its lateral surface is not insulated: at each time, there is an energy loss in the rod through its lateral surface. We assume that the energy lost by the system around a point x at time t equals $\alpha u(t,x)$ per unit area, where $\alpha > 0$.

(1) Show that $u(t,x)$ solves the following PDE:

$$c\rho\frac{\partial u}{\partial t} - \kappa\frac{\partial^2 u}{\partial x^2} + \frac{\alpha P}{A}u = 0, \tag{4}$$

with boundary conditions:

$$\forall t > 0,\ \ u(t,0) = 0, \quad \text{and } u(t,\ell) = 0. \tag{5}$$

(2) For the sake of simplification, we thenceforth assume that $c\rho = 1$ and $\frac{P}{A} = 1$. Find the equilibrium temperature $u_{eq}(x)$ in the rod, if any.
(3) We use the method of separation of variables to solve (4-5). Write down the ODE satisfied by the temporal part $T(t)$ and spatial part $X(x)$ of a separated solution $u(t,x) = T(t)X(x)$. In particular, show that there exists $\lambda \in \mathbb{R}$ such that $X(x)$ satisfies:

$$\begin{cases} -X''(x) + \frac{\alpha}{\kappa}X(x) = \lambda X(x) & \text{for } x \in (0,\ell) \\ \quad X(0) = 0,\ X(1) = 0 \end{cases}. \tag{6}$$

(4) Is $\lambda = \frac{\alpha}{\kappa}$ an eigenvalue of the problem?
(5) Search for the eigenvalues λ of the form $\lambda = \frac{\alpha}{\kappa} + \beta^2$, $\beta > 0$.
(6) Search for the eigenvalues λ of the form $\lambda = \frac{\alpha}{\kappa} - \beta^2$, $\beta > 0$.
(7) Solve the temporal equation associated to each eigenvalue λ and write down the general series expansion for solutions to the system (4 - 5).
(8) Do you retrieve the result of Question 1 from the obtained series for solutions $u(t,x)$ to (4 - 5)?