Elementary partial differential equations: homework 4

Assigned 02/20/2014, due 02/27/2014.

Exercise 1

This exercise is reprinted from [Strauss], §2.2, Exercise 2.

Let $u(t,x)$ be a solution to the wave equation:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0,$$

where the velocity c has been set to 1 for simplicity, and define the *energy density* $e(t,x)$ and the *momentum* $p(t,x)$ as:

$$e = \frac{1}{2}\left(\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right), \;\; p = \frac{\partial u}{\partial t}\frac{\partial u}{\partial x}.$$

(1) Show the following relations:

$$\frac{\partial e}{\partial t} = \frac{\partial p}{\partial x}, \;\; \frac{\partial p}{\partial t} = \frac{\partial e}{\partial x}.$$

(2) Show that both $e(t,x)$ and $p(t,x)$ are also solutions to the wave equation.

Exercise 2

This exercise is reprinted from [Strauss], §2.2, Exercise 3.

Let $u(t,x)$ be a solution to the wave equation:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0.$$

(1) Show that any translate $u(t, x-y)$, where $y \in \mathbb{R}$ is fixed, is again a solution to the wave equation.
(2) Show that the derivative $\frac{\partial u}{\partial x}$ is also a solution to the wave equation.
(3) Show that any dilation $u(at, ax)$, where $a \in \mathbb{R}$ is a given constant, is also a solution to the wave equation.

Exercise 3

This exercise is partly reprinted from [Strauss], §2.1, Exercises 5.

The purpose of this exercise is to understand the behavior of the solution $u(t,x)$ of the one-dimensional wave equation in a particular physical context. Let $u(t,x)$, $t > 0$, $x \in \mathbb{R}$, be the unique solution to the problem:

$$(1) \qquad \begin{cases} \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 & \text{for } t > 0,\ x \in \mathbb{R}, \\ u(0,x) = \phi(x) & \text{for } x \in \mathbb{R}, \\ \frac{\partial u}{\partial t}(0,x) = \psi(x) & \text{for } x \in \mathbb{R}, \end{cases}$$

where $c > 0$ stands for the velocity, and where the initial position $\phi(x)$ and the initial velocity $\psi(x)$ of the considered string are respectively given by:

$$\forall x \in \mathbb{R}, \;\; \phi(x) = 0, \text{ and } \psi(x) = \begin{cases} 0 & \text{if } |x| > a \\ 1 & \text{if } |x| \leq a, \end{cases}$$

$a > 0$ being a fixed parameter.

(1) This situation is sometimes referred to as the *hammer blow*. Can you explain why?

(2) By using the exact formula for the unique solution to (1), show that:

$$\forall t > 0,\ \forall x \in \mathbb{R},\ \ u(t,x) = \frac{1}{2c}\text{len}\Big((x-ct, x+ct) \cap (-a,a)\Big),$$

where $\text{len}(I) = d - c$ stands for the length of an interval $I = (c,d)$.

(3) Draw the string profile (i.e. the values of u versus x) at the following particular times:
   (a) At time $t = 0$,
   (b) At time $t = \frac{a}{2c}$,
   (c) At time $t = \frac{a}{c}$,
   (d) At time $t = \frac{3a}{2c}$,
   (e) At time $t = \frac{2a}{c}$,
   (f) At time $t = \frac{5a}{c}$.

Warning: Pay attention to reporting on the figures all the relevant values of x and $u(t,x)$.

Exercise 4

This exercise is partly reprinted from [Strauss], §2.1, Exercise 11.

The purpose of this exercise is to find the general solution to the PDE:

$$3\frac{\partial^2 u}{\partial t^2} + 10\frac{\partial u}{\partial x \partial t} + 3\frac{\partial u}{\partial x^2} = \sin(x+t), \tag{2}$$

by using a factorization method close to the one seen during the lectures.

(1) Show that (2) can be rewritten as:

$$\left(\frac{\partial}{\partial t} + \frac{5}{3}\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \frac{5}{3}\frac{\partial}{\partial x}\right)u - \frac{16}{9}\frac{\partial^2 u}{\partial x^2} = \frac{1}{3}\sin(x+t). \tag{3}$$

(2) Find an adequate new set of variables (ξ, η) such that (3) becomes:

$$\frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \eta^2} = \frac{1}{3}\sin\left(\frac{8}{3}\xi + \frac{4}{3}\eta\right). \tag{4}$$

(3) Find a particular solution to (4) under the form $(\xi,\eta) \mapsto A\sin\left(\frac{8}{3}\xi + \frac{4}{3}\eta\right)$, for some constant A to be determined.

(4) Conclude as for the form of the general solution to (2).

Exercise 5

This exercise is partly reprinted from [Strauss], §2.1, Exercise 10.

By using the factorization method seen during the lectures for solving the wave equation, solve the PDE:

$$\frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial x \partial t} - 3\frac{\partial u}{\partial x^2} = 0,$$

with the two initial conditions:

$$\forall x \in \mathbb{R},\ \ u(0,x) = x^2,\ \ \frac{\partial u}{\partial t}(0,x) = e^x.$$

Exercise 6

This exercise is optional, and is reprinted from [Strauss], §2.2, exercise 1.

Use the energy conservation of the wave equation to prove that the only solution $u(t,x)$ to:

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0,$$

with initial conditions:

$$\forall x \in \mathbb{R},\ \ u(0,x) = 0,\ \ \frac{\partial u}{\partial t}(0,x) = 0$$

is $u \equiv 0$.