

## Elementary partial differential equations: homework 2

Assigned 02/04/2014, due 02/11/2014.

This homework is composed of an exercise, plus a problem of three parts, which can be dealt with in a completely independent fashion.

### Exercise 1

*This exercise is reprinted from [Haberman], §1.2, Exercise 9.*

Consider a three dimensional rod oriented along the  $x$ -axis, with cross-sectional area  $A$  and lateral perimeter  $P$  (see Figure 1). The rod is very thin, and lies in the region  $(0 < x < L)$ , so that we assume that its temperature  $u$  only depends on the time  $t$  and on  $x$ . It does not contain thermal sources, but its lateral surface is not insulated. The density of the rod is denoted as  $\rho$ , its specific heat as  $c$ , the Fourier constant as  $\kappa$ , and these quantities may vary along  $x$  (i.e. the rod is possibly inhomogeneous).

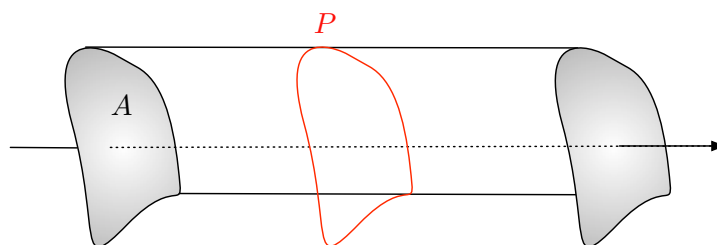


FIGURE 1. Setting for Exercise 1.

- (1) Assume that the heat energy flowing *out* of the lateral surface of the rod, per unit length and per unit time is  $w(t, x)$  (that is, the energy flowing out at  $x$  and at time  $t$  is  $Pw(t, x)$ ). By reproducing the derivation of the heat equation studied during the lectures, find the PDE satisfied by the temperature  $u(t, x)$  in the rod.
- (2) Assume that  $w(t, x)$  is proportional to the temperature difference between that  $u(t, x)$  of the rod, and a known temperature  $\gamma(t, x)$  of the outer medium, i.e.:

$$w(t, x) = h(x)(u(t, x) - \gamma(t, x)).$$

What is the sign of  $h$  according to physical intuition? Show that, in this case, the PDE satisfied by  $u$  reads:

$$c\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( \kappa \frac{\partial u}{\partial x} \right) - \frac{P}{A} h(x)(u(t, x) - \gamma(t, x)).$$

- (3) Compare this last equation to the heat equation considered during the lectures. What is similar, different?
- (4) We now assume that the cross-section of the rod is a disk with radius  $r$ , that the outside temperature is  $\gamma(t, x) = 0$ , and that the rod is *homogeneous*. Write the new corresponding equation.
- (5) Still in the context of (4), we assume that the temperature in the rod is *uniform*, i.e only depends on  $t$ ,  $u(t, x) = u(t)$ . Determine  $u(t)$ , provided the initial condition reads  $u(0) = u_0$ , for some known function  $u_0$ .

**Problem**

This exercise is partly reprinted from [Haberman], §1.5, exercises 3, 5, 9.

The three parts of this problem are completely independent, and are devoted to the study of the solution  $u(t, x, y)$  of the heat equation in a plane domain  $\Omega$ . This equation reads:

$$(1) \quad c\rho \frac{\partial u}{\partial t} = \kappa \Delta u,$$

where the coefficients  $c, \rho$  and  $\kappa$  (corresponding respectively to the specific heat per unit of mass, the density and the Fourier coefficient) are assumed constant. More particularly, we will get interested in the case where the solution is *radial*.

**Part 1.** The goal of this first part is to reach an expression for the heat equation in the radial case by using the expression of the Laplace operator in polar coordinates. In the plane  $\mathbb{R}^2$ , the usual Cartesian coordinates are denoted as  $(x, y)$ , and the *polar coordinates*  $(r, \theta)$  are defined by:

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} .$$

(1) Show that:

$$\frac{\partial r}{\partial x} = \cos \theta, \quad \frac{\partial r}{\partial y} = \sin \theta, \quad \frac{\partial \theta}{\partial x} = \frac{-\sin \theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{\cos \theta}{r}.$$

[Hint: use the fact that  $r^2 = x^2 + y^2$ .]

- (2) Let  $u(x, y)$  be any differentiable function. Express  $\frac{\partial u}{\partial x}$ ,  $\frac{\partial u}{\partial y}$  and then  $\frac{\partial^2 u}{\partial x^2}$ ,  $\frac{\partial^2 u}{\partial y^2}$  in terms of the derivatives of  $u$  with respect to  $\theta$ .
- (3) Show that  $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$  equals to, in polar coordinates:

$$\Delta u = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}.$$

- (4) Suppose that the solution  $u(t, x, y) = u(t, r, \theta)$  is *radial*, i.e. it does not depend on  $\theta$ . Write the particular form of the heat equation (1) in this context.

**Part 2.** We thenceforth assume that the considered domain  $\Omega$  is an annulus, i.e.:

$$\Omega := \{(x, y) \in \mathbb{R}^2, a < r < b\},$$

where  $r = \sqrt{x^2 + y^2}$  and  $0 < a < b$  are two constants. We assume that the temperature  $u$  in the annulus is radial, i.e.  $u = u(t, r)$ , and proceed as in the lectures to derive the PDE satisfied by  $u$ .

- (1) Show that the total energy concentrated in a slice  $(r_0, r_0 + h)$  of the annulus at time  $t$  is:

$$H(t) = 2\pi \int_{r_0}^{r_0+h} c\rho u r \, dr.$$

- (2) Show that the heat energy *entering* the domain at  $r = r_0 + h$  is:

$$+2\pi(r_0 + h)\kappa \left. \frac{\partial u}{\partial r} \right|_{r=r_0+h} .$$

A similar result holds at  $r = r_0$ .

- (3) Use the previous two questions to derive the circularly symmetric form of the heat equation without sources:

$$\frac{\partial u}{\partial t} = \frac{\kappa}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right),$$

and make sure you end up with the same expression as the result of Part 1.

**Part 3.** We still consider that  $\Omega$  is the annulus of Part 2, and assume that the temperature  $u(t, r)$  reaches an equilibrium state, i.e.  $u(t, r) = u(r)$  (this is the temperature in the annulus after a very long time). Then,  $u(r)$  is a solution to the corresponding *stationnary heat equation*:

$$\frac{k}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) = 0.$$

- (1) Determine  $u(r)$  when the annulus is submitted to a Dirichlet boundary condition at both its inner and outer radii:

$$u(a) = T_1, \quad u(b) = T_2,$$

for some fixed temperatures  $T_1, T_2$ .

- (2) Do the same if the annulus is submitted to a Dirichlet boundary condition  $u(a) = T_1$  at its inner radius, and is insulated at the outer radius (homogeneous Neumann conditions).