Elementary partial differential equations: homework 1

Assigned 01/28/2014, due 02/04/2014.

This homework is composed of five exercises, plus an additional one, which is optional and should only be considered provided the former ones have already been well addressed and understood.

Exercise 1

This exercise is partly reprinted from [Strauss], §1.1, exercise 3.

For each of the following partial differential equations, (1) write the associated differential operator, (2) state the order of the equation, and (3) whether it is linear, homogeneous. When it is nonlinear, justify.

(1) $\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = x^4$, where $u \equiv u(t,x)$.
(2) $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + xu = 0$, where $u \equiv u(t,x)$.
(3) $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x} = 0$, where $u \equiv u(t,x)$.
(4) $\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + x^2 = 0$, where $u \equiv u(t,x)$.
(5) $i\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + \frac{u}{x} = 0$, where $u \equiv u(t,x)$.
(6) $\frac{\partial u}{\partial x}\left(1 + \left(\frac{\partial u}{\partial x}\right)^2\right)^{-\frac{1}{2}} + \frac{\partial u}{\partial y}\left(1 + \left(\frac{\partial u}{\partial y}\right)^2\right)^{-\frac{1}{2}} = 0$, where $u \equiv u(x,y)$.
(7) $\frac{\partial u}{\partial x} + e^y\frac{\partial u}{\partial y} = 0$, where $u \equiv u(x,y)$.
(8) $i\frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} + \sqrt{1 + \frac{\partial u}{\partial x}} = 0$, where $u \equiv u(t,x)$.

Exercise 2

This exercise is partly reprinted from [Strauss], §1.1, exercise 11.

(1) Verify that, for all pairs (f,g) of differentiable functions $f,g : \mathbb{R} \to \mathbb{R}$, the function $u(x,y) = f(x)g(y)$ is a solution to the PDE:
$$u\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}.$$
(2) Find all the solutions to the PDE
$$\frac{\partial u}{\partial x} + u = 0,$$
where the unknown function u is a function of two variables $u \equiv u(x,y)$. ‘How many’ solutions do you end up with ?
(3) Same question as (2), with the additional ‘boundary condition’ $u(0,y) = y^3$ for all $y \in \mathbb{R}$.

Exercise 3

This exercise is partly reprinted from [Strauss], §1.2, exercise 4.

(1) Solve the partial differential equation
$$-(1+x^2)\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$
by using the method of characteristics, where the unknown function u is a function of two variables $u \equiv u(x,y)$.
(2) Draw several characteristic curves.

(3) Verify that the solutions you end up with indeed solve the considered PDE, by computing its partial derivatives, and evaluating

$$\mathcal{L}(u) = -(1+x^2)\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}.$$

Exercise 4

This exercise is reprinted from [Haberman], §12.2, exercise 5.

Solve the partial differential equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = e^{2x}$$

of an unknown function of two variables $u \equiv u(t,x)$, where $c \in \mathbb{R}$ is a fixed parameter, and with the additional 'initial condition' $u(0,x) = f(x)$, where f is a given function. Draw several characteristic curves.

Exercise 5

This exercise is partly reprinted from [Strauss], §1.2, exercise 10.

(1) Solve the ordinary differential equation

$$\frac{dx}{ds}(s) + x(s) = e^{3s+2c_0},$$

where $c \in \mathbb{R}$ is an arbitrary constant.

(2) Deduce from (1) the solution to the partial differential equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + u = e^{x+2y}$$

of an unknown function $u \equiv u(x,y)$ of two variables, with the 'boundary condition' $u(x,0) = 0$.

Exercise 6

This exercise is optional, and is reprinted from [Strauss], §1.2, exercise 7.

(1) Solve the partial differential equation

$$y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial x} = 0$$

of an unknown function $u \equiv u(x,y)$ of two variables, with the 'boundary condition' $u(0,y) = y^3$.

(2) In which region of the xy plane is the solution uniquely determined ?