Advanced Calculus I: Workshop 9

Exercise 1

Let $a < b$ be two real numbers, and $f : [a, b] \to [a, b]$ be a continuous function. Show that f has a *fixed point* x_0 in $[a, b]$, i.e. that there exists $x_0 \in [a, b]$ such that:

$$f(x_0) = x_0.$$

Exercise 2

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that:

$$\lim_{x\to-\infty} f(x) = 0, \ \ \lim_{x\to+\infty} f(x) = 0, \ \ f(0) = 1.$$

(1) Make a drawing of the situation.
(2) Express in terms of quantifiers the meaning of the properties $\lim_{x\to-\infty} f(x) = 0$, and $\lim_{x\to+\infty} f(x) = 0$.
(3) Infer that there exists a real number $A > 0$ such that:

$$\forall x \in \mathbb{R} \text{ s.t. } |x| > A, \ \ |f(x)| < \frac{1}{2}.$$

(4) Show that f is bounded.
[Hint: use the fact that a continuous function on a compact set is bounded.]
(5) Show that f has a maximum over $\mathbb{R}$, i.e. that there exists a point $x_0 \in \mathbb{R}$ such that:

$$\forall x \in \mathbb{R}, \ \ f(x) \le f(x_0).$$