Advanced Calculus I: Workshop 6

Exercise 1

Recall that the *integer part* $[x]$ of a real number x is defined as the unique integer $n \in \mathbb{Z}$ such that:

$$n \leq x < n+1.$$

(1) Draw the graph of the function $x \mapsto [x]$.
(2) Let $f : x \mapsto x - [x]$. Draw the graph of f.
(3) Determine the real numbers x_0 at which f has a limit and those at which it does not, and prove your assertions.

Exercise 2

Let $D \subset \mathbb{R}$, let $f : D \to \mathbb{R}$ be a function, and x_0 be an accumulation point of D. Prove that f has a limit at x_0 if and only if, for any $\varepsilon > 0$, there exists a neighborhood Q of x_0 such that: $\forall x, y \in D \cap Q$, $|f(x) - f(y)| < \varepsilon$.

[Hint: the $\Rightarrow$ part is an application of the definition of the limit of a function; for the converse part $\Leftarrow$, argue by contradiction, assuming that f does not have a limit at x_0. Prove that there exists then a sequence $x_n \to x_0$, $x_n \neq x_0$ such that $\{f(x_n)\}$ does not converge, and use the Cauchy criterion.]