Advanced Calculus I: Revisions for Midterm 2

Exercise 0:

(1) Let D be a subset of $\mathbb{R}$, and $x_0 \in D$. Let $f : D \to \mathbb{R}$ be continuous at x_0, and such that $f(x_0) \neq 0$. Show that there exists a neighborhood Q of x_0 such that:

$$\forall x \in D \cap Q, \ \ f(x) \neq 0.$$

(2) Define *precisely* the notion of a uniformly continuous function f on a subset $D \subset \mathbb{R}$.

(3) Let $D \subset \mathbb{R}$, and $x_0 \in D$ be an accumulation point of D. Prove that, if $f, g : D \to \mathbb{R}$ are two functions which are differentiable at x_0, then so is their product fg, and provide its derivative.

(4) Let $D \subset \mathbb{R}$, $x_0 \in D$, and consider a function $f : D \to \mathbb{R}$. Express, in terms of quantifiers, what it means for f not to be continuous at x_0.

(5) Recall the definition of a compact set; then, state the Heine-Borel theorem.

(6) Let $D \subset \mathbb{R}$, and $f : D \to \mathbb{R}$ be a function; prove that, if f is uniformly continuous on D, then it is continuous at any point $x_0 \in D$.

(7) State Rolle's theorem.

(8) State the intermediate-value theorem.

(9) State the mean-value theorem.

(10) State the theorem of sequential characterization of the limit of a function at a point.

Exercise 1:

(1) State the theorem around sequences of elements lying in a compact set $K \subset \mathbb{R}$.

(2) Show that the interval $A = [0, 2)$ is not compact by finding a sequence $\{x_n\}$ of elements of A which does not satisfy the criterion stated in (1).

(3) State the definition of a compact subset of $\mathbb{R}$.

(4) Show that this interval is not compact by using only the definition of compactness.

(5) Show that this interval is not compact by using the Heine-Borel theorem.

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Exercise 2:

f, g continuous on $[0, 1]$, such that:

$$0 < f(x) < g(x).$$

Show that $h = f/g$ is well-defined and continuous. Show that its image is in $(0, 1)$. Show that there exist m, M st

$$mg \leq f \leq Mg.$$

Exercise 3:

Let $a < b$ be two real numbers, and let $f : [a, b] \to \mathbb{R}$ be a function which is continuous on $[a, b]$ and differentiable on (a, b). Assume in addition that there exists a real number $M > 0$ such that, for all $x \in (a, b)$:

$$|f'(x)| \leq M.$$

Show that, for all $x, y \in [a, b]$, one has:

$$|f(x) - f(y)| \leq M|x - y|.$$

Exercise 4:

(1) Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be the function defined by:

$$\forall x \neq 0, \ \ f(x) = \frac{\sin(x)}{x}.$$

Show that f has a limit at 0 and calculate this limit.

(2) Let $g : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be the function defined by:

$$\forall x \neq 0, \;\; f(x) = \frac{\cos(x) - 1}{x}.$$

Show that f has a limit at 0 and calculate this limit.

Exercise 5:

Let $f : [0, 2] \to \mathbb{R}$ be the function defined by:

$$f(x) = |x - 1|.$$

Show that $f(0) = f(2)$, but that there does not exists any $c \in [0, 2]$ such that $f'(c) = 0$. Why does it not come in contradiction with Rolle's theorem?

Exercise 6:

By using the mean-value theorem, show that:

(1) For any real numbers $0 < a < b$, one has the inequality:

$$\frac{1}{b} < \frac{\log(b) - \log(a)}{b - a} < \frac{1}{a}.$$

(2) For all $x > 0$, one has:

$$\frac{1}{x+1} < \log(1 + x) - \log(x) < \frac{1}{x}.$$

Exercise 7:

The purpose of this exercise is to provide an alternative proof of the Heine theorem. Let $K \subset \mathbb{R}$ be a compact set, and let $f : K \to \mathbb{R}$ be a continuous function. The proof goes by a contradiction argument.

(1) Negate the definition of uniform continuity for f.

(2) Show that, if f is not uniformly continuous on K, then there exist $\varepsilon > 0$, as well as two sequences $\{x_n\}_{n\in\mathbb{N}^*}$ and $\{y_n\}_{n\in\mathbb{N}^*}$ of elements of K which satisfy the following properties:

$$\forall n \in \mathbb{N}^*, \;\; |x_n - y_n| < \frac{1}{n}, \text{ and } |f(x_n) - f(y_n)| > \varepsilon.$$

(3) Show that there exist two subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ of $\{x_n\}_{n\in\mathbb{N}^*}$ and $\{y_n\}_{n\in\mathbb{N}^*}$ respectively, which converge to a common limit $\alpha \in K$.

(4) End the proof by obtaining a contradiction between this fact and the properties of Question (2).

Exercise 8:

(1) Is the function $f(x) = \frac{1}{x^2}$ uniformly continuous on $(0, +\infty)$? In any case, prove your answer.

(2) Is it true that, if $f : [0, 1] \to \mathbb{R}$ is a continuous function such that, for all $x \in [0, 1]$, $f(x) \neq 0$, then $g(x) = \frac{1}{f(x)}$ is a uniformly continuous function? If your answer is yes, prove it; else, provide a counterexample.

Exercise 9:

Let I be an interval of $\mathbb{R}$, $f, g : I \to \mathbb{R}$ be two continuous functions such that:

$$\forall x \in I, \;\; |f(x)| = |g(x)| \neq 0.$$

Show that, either

$$\forall x \in I, \;\; f(x) = g(x),$$

or

$$\forall x \in I, \;\; f(x) = -g(x).$$

Exercise 10:

Let $a < b$ be two real numbers, and $f : [a, b] \to \mathbb{R}$ be a function such that f is differentiable on (a, b), and:

$$\forall x \in [a, b], \;\; f(x) > 0.$$

Show that there exists $c \in (a, b)$ such that:

$$\frac{f(a)}{f(b)} = e^{(b-a)\frac{f'(c)}{f(c)}}.$$