Advanced Calculus I: Homework 8

Assigned 10/30/2014, due 11/06/2014.

In all the exercises of this sheet, you may use as a result of the lectures the following statement, which stems from Exercise 2 of the 8^{th} workshop session:

- Any union of open subsets of $\mathbb{R}$ is open.
- An intersection of a *finite* number of open subsets of $\mathbb{R}$ is open.
- Any intersection of closed subsets of $\mathbb{R}$ is closed.
- A union of a *finite* number of closed subsets of $\mathbb{R}$ is closed.

Exercise 1

Say (with a brief justification) if each of the following subsets of $\mathbb{R}$ is open, closed, bounded and/or compact.

(1)	The empty set $\emptyset$	(2)	$\mathbb{R}$
(3)	The set $\mathbb{Q}$ of rational numbers	(4)	$\mathbb{Q} \cap [0,1]$.
(5)	The set $\mathbb{Z}$ of integers	(6)	$[0,4)$
(7)	$[0,5] \cup \{7\}$	(8)	$(0,1) \cup (2,3)$

Exercise 2

The purpose of this exercise is to prove with two different methods that the set $(1,2)$ is not compact.

(1) Show it directly, as a consequence of the Heine-Borel theorem.
(2) Negate the *definition* of compacity by constructing an open cover $\{O_i\}_{i\in I}$ of $(1,2)$ which does not have a finite subcover.

Exercise 3 *(Partially reprinted from Ex. 23 p. 105 in [Gaughan]).*

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *periodic* if there exists a real number $T > 0$ such that:

$$\forall x \in \mathbb{R}, \ f(x+T) = f(x).$$

(1) Give an example of a *non constant* periodic function and draw it.
(2) Show that if $f : \mathbb{R} \to \mathbb{R}$ is periodic and continuous, then it is uniformly continuous.
[Hint: start by showing that f is uniformly continuous on $[0,T]$.]

Exercise 4 *(Reprinted from Ex. 25 p. 104 in [Gaughan]).*

Provide an example of two sets $A, B \subset \mathbb{R}$ and of a function $f : A \cup B \to \mathbb{R}$ which is continuous on $A \cup B$, uniformly continuous on A and on B, but which is not uniformly continuous on $A \cup B$.

Draw your example.

Exercise 5

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, which has finite limits a and b at $-\infty$ and $+\infty$ respectively. The purpose of this exercise is to show that f is uniformly continuous on $\mathbb{R}$.

(1) Explain, e.g. with a drawing, why, intuitively, this property should hold.
(2) Recall, by means of quantifiers, what it means for f to have finite limits a and b at $-\infty$ and $+\infty$.
(3) By using the fact that f has limit a at $-\infty$, prove that for any $\varepsilon > 0$, there exists $A < 0$ such that:

$$\forall x \in (-\infty, A), \ |f(x) - a| < \varepsilon.$$

Similarly, we admit that for any $\varepsilon > 0$, there exists $B > 0$ such that:

$$\forall x \in (B, +\infty), \ |f(x) - b| < \varepsilon.$$

(4) Show that f is uniformly continuous on $[A-1, B+1]$.
(5) Using the results of (3) and (4), prove that f is uniformly continuous on $\mathbb{R}$.

Exercise 6

The purpose of this exercise is to study the *closure* $\overline{D}$ of an arbitrary subset $D \subset \mathbb{R}$, which is defined as:

$$\overline{D} = \bigcap_{\substack{F \text{ is closed} \\ D \subset F}} F.$$

(1) Show that, for any subset $D \subset \mathbb{R}$, the closure $\overline{D}$ is a closed set.

(2) Show that it is the *smallest* (for inclusion) closed set of $\mathbb{R}$ which contains D, i.e.

$$\text{if } F \text{ is any closed set such that } D \subset F, \text{ then } \overline{D} \subset F.$$

(3) Show that D is closed if and only if $D = \overline{D}$.

(4) *(This question is more difficult and is optional):* Let D' be the set of accumulation points of D. Show, by double inclusion, that $\overline{D} = D \cup D'$.

[Hint: Use double inclusion; observe that, to prove $\overline{D} \subset D \cup D'$, it is enough to show that $D \cup D'$ is a closed set.]