Advanced Calculus I: Homework 7

Assigned 10/23/2014, due 10/30/2014.

Exercise 1

Specify the domain of each of the following functions, then draw it, and say at which points it is continuous:

(1)	$x \mapsto \lvert x \rvert$	(2)	$x \mapsto x^2 + 3x + 1$
(3)	$x \mapsto \sqrt{x}$	(4)	$x \mapsto \frac{1}{x+1}$
(5)	$x \mapsto \sin^2(x) + 3\sin(x) + \cos(x)$	(6)	$x \mapsto \begin{cases} 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{Q} \end{cases}$

Exercise 2

Let D be a subset of $\mathbb{R}$, and $E \subset D$. Let $f : D \to \mathbb{R}$ be a function. Let $g : E \to \mathbb{R}$ be the *restriction* of f to E, i.e.:

$$\forall x \in E, \ \ g(x) = f(x).$$

(1) Show that, if f is continuous, then so is g.

(2) Does the converse property necessarily hold? If your answer is yes, prove it, else provide a counterexample.

Exercise 3

Let $a < b$ be two real numbers, and let $f : (a, b) \to \mathbb{R}$ be a function. One says that f is *Lipschitz* if there exists a real number $\alpha > 0$ such that:

$$\forall x, y \in (a, b), \ \ |f(x) - f(y)| \leq \alpha |x - y|.$$

(1) Give an example of a Lipschitz function, defined on some interval (a, b) of your choice; give explicitely the value of the constant α.

(2) Show that a Lipschitz function $f : (a, b) \to \mathbb{R}$ is continuous.

Exercise 4 *(Reprinted from Ex. 4 p. 104 in [Gaughan]).*

Let D be a subset of $\mathbb{R}$, $f : D \to \mathbb{R}$ be a function, and let $x_0 \in D$. Show that, if x_0 is not an accumulation point of D, then f is automatically continuous at x_0.

Exercise 5

Let D be a subset of $\mathbb{R}$, $f, g : D \to \mathbb{R}$ be two functions, and let $x_0 \in D$. We assume that f and g are continuous at x_0. Show that, if $f(x_0) > g(x_0)$, then there exists a neighborhood Q of x_0 such that:

$$\forall x \in Q \cap D, \ \ f(x) > g(x).$$

[Hint: it may be worth drawing the situation...]

Exercise 6

Let $f, g : \mathbb{R} \to \mathbb{R}$ be two continuous functions over $\mathbb{R}$, such that, for any rational number $a \in \mathbb{Q}$, one has $f(a) = g(a)$.

(1) Let x be an arbitrary real number. Why does there exist a sequence $\{r_n\}$ of rational numbers such that $r_n \to x$?

(2) By using (1) and the characterization of the continuity of functions in terms of sequences, show that, for any real number x, $f(x) = g(x)$.