Advanced Calculus I: Homework 4

Assigned 10/02/2014, due 10/09/2014.

Exercise 1 *(Reprinted from Ex. 39 p. 57 in [Gaughan]).* Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences of real numbers, which converge to the same limit $\ell \in \mathbb{R}$. Define a new sequence $(c_n)_{n\in\mathbb{N}}$ as:

$$\forall n \in \mathbb{N},\ \ c_{2n} = a_n,\ \text{ and } c_{2n+1} = b_n.$$

Show that (c_n) also converges to ℓ.

Exercise 2 Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.

(1) In this question, we assume that the two subsequences (a_{2n}) and (a_{2n+1}) converge to the same limit $\ell \in \mathbb{R}$. Show that $a_n \to \ell$.

(2) Construct an example of a sequence (a_n) such that (a_{2n}) and (a_{2n+1}) both converge, whereas (a_n) does not.

(3) *This question is slightly more difficult, and is not required in the homework*. In this question, we assume that the three subsequences (a_{2n}) and (a_{2n+1}) and (a_{3n}) converge. Show that they converge to a common limit $\ell \in \mathbb{R}$, and that $a_n \to \ell$.

Exercise 3 *(Partially reprinted from Ex. 37 p. 57 in [Gaughan])*

(1) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers which is decreasing and bounded. Show that (a_n) is convergent.

(2) Express in terms of quantifiers what it means for a sequence (a_n) *not* to be bounded.

(3) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers which is increasing, but not bounded. Show that (a_n) goes to $+\infty$.

Exercise 4 *(Partially reprinted from Ex. 40 p. 57 in [Gaughan])*

Define recursively the sequence $(a_n)_{n\in\mathbb{N}}$ as:

- $a_0 = 6$.
- $\forall n \in \mathbb{N},\ \ a_{n+1} = \sqrt{6 + a_n}$.

(1) Show that, for all $n \in \mathbb{N}$, $3 \le a_n \le 6$.

(2) Show that (a_n) is decreasing.

[Hint: Calculate $(a_{n+1} - a_n)$ and study its sign, using the result of Question (1).]

(3) Conclude that it is convergent, and calculate its limit.

Exercise 5

Give an example of a Cauchy sequence $(a_n)_{n\in\mathbb{N}}$, whose terms belong to $(0, 1]$, which does not converge to a real number $\ell \in (0, 1]$.

Exercise 6

(1) Let $r \in \mathbb{R}$ be such that $0 < r < 1$. Show that:

$$\forall n \in \mathbb{N},\ \ 1 + r + ... + r^n = \frac{1 - r^{n+1}}{1 - r}.$$

(2) Conclude that

$$\forall n, k \in \mathbb{N},\ \ r^n + r^{n+1} + ... + r^{n+k} \le \frac{r^n}{1 - r}.$$

Let now $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that there exists a real $r > 0$ such that:

$$\forall n \in \mathbb{N}^*,\ \ |a_{n+1} - a_n| \le r|a_n - a_{n-1}|$$

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(3) Show that, for all $n \in \mathbb{N}$,

$$|a_{n+1} - a_n| \leq r^n |a_1 - a_0|.$$

(4) By using the result of question (2), and the triangle inequality, show that (a_n) is a Cauchy sequence.