Advanced Calculus I: Homework 10

Assigned 11/13/2014, due 11/20/2014.

Exercise 1

Let C be a non empty, bounded and connected subset of $\mathbb{R}$. Let $a = \inf(C)$ and $b = \sup(C)$. We assume that $a \in C$, but $b \notin C$.

(1) Why do $a = \inf(C)$ and $b = \sup(C)$ exist?
(2) Show that $C = [a, b)$.

Exercise 2

In each of the following situations, identify the domain of definition of the function f, and the set of points where it is differentiable. Then, calculate $f'(x)$ wherever it makes sense.

(1) $f(x) = |3x + x^2|$ (2) $f(x) = x^2 \tan(x)$
(3) $f(x) = \frac{3x-2}{2x-3}$ (4) $f(x) = \frac{x^2+2}{x^2-1}$.
(5) $f(x) = \frac{\cos(x)}{1+2\sin(x)}$ (6) $f(x) = \frac{e^x \log(x)}{x^2+2x^3}$

Exercise 3 *(Partially reprinted from Ex. 12, 13 p. 130 in [Gaughan]).*

(1) Let $f : [a, b] \to [c, d]$, $g : [c, d] \to \mathbb{R}$ be two differentiable functions, such that f' and g' are also differentiable. By using the chain-rule, show that $(g \circ f)'$ is differentiable on $[a, b]$ and calculate its derivative.
(2) Let $f : [a, b] \to [c, d]$, $g : [c, d] \to [p, q]$ and $h : [p, q] \to \mathbb{R}$ be three functions. Let $x_0 \in [a, b]$ be a point such that f is differentiable at x_0, g is differentiable at $f(x_0)$ and h is differentiable at $g \circ f(x_0)$. By using the chain rule, show that $h \circ (g \circ f)$ is differentiable at x_0, and express its derivative in terms of those of f, g, h.

Exercise 4

Let $a < b$ be two real numbers, and let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(a) = f(b)$. Define the function $g : \left[a, \frac{a+b}{2}\right] \to \mathbb{R}$ as:

$$g(t) = f\left(t + \frac{b-a}{2}\right) - f(t).$$

(1) Show that there exists a real number $t_0 \in \left[a, \frac{a+b}{2}\right]$ such that $g(t_0) = 0$.
(2) *Application:* A person travels 4 miles within an hour. Show that there exists an interval of time of half an hour during which it travels exactly 2 miles.

Exercise 5

Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and $x_0 \in \mathbb{R}$.

(1) Recall the definition of the differentiability of f at x_0.
(2) Prove that, if f is differentiable at x_0, the *two-sided rate of change* $R(t)$, defined by:

$$\forall t \neq 0, \ \ R(t) = \frac{f(x_0 + t) - f(x_0 - t)}{2t}$$

has limit $f'(x_0)$ at x_0.
(3) Assume now that f is a function such that $\lim_{t \to 0} R(t)$ exists, and let $\ell \in \mathbb{R}$ be that limit. Is it necessarily true that f is differentiable at x_0? If your answer is yes, prove it; else, provide a counterexample.

Exercise 6

Let $n \in \mathbb{N}$ be fixed; define the function $f_n : [0, +\infty) \to \mathbb{R}$ by:

$$\forall x \in [0, +\infty)\,,\ \ f_n(x) = \log(1 + x^n) + x - 1.$$

(1) Show that there exists a real number $x \in [0, 1]$ such that $f_n(x) = 0$.

(2) Show that f_n is a strictly increasing function.
[Hint: show that f_n is differentiable, and study the sign of its derivative.]

(3) Deduce from your answers to (1) and (2) that there exists a *unique* real number $x \in [0, 1]$ such that $f_n(x) = 0$.