Advanced Calculus I: Homework 1

Assigned 09/09/2014, due 16/09/2014.

Notations:

- In the whole sheet of exercises, unless otherwise specified, every mentioned function $f : A \to B$ between two sets A and B is implicitly considered to be defined on the whole set A (i.e. its domain is A).
- If $a < b$ are two real numbers, $(a, b) = \{x \in \mathbb{R}, a < x < b\}$ (resp. $[a, b] = \{x \in \mathbb{R}, a \leq x \leq b\}$) stands for the *open* (resp. *closed*) interval with bounds a and b.

Exercise 1: *(Modified version of Ex.* 10 *p.* 28 *in [Gaughan]).*

Describe each of the following sets as the empty set, $\mathbb{R}$ as a whole, or as an appropriate combination of real intervals:

(1) $\bigcap_{n=1}^{\infty} \left[0, \frac{1}{n}\right]$,

(2) $\bigcup_{n=1}^{\infty} (-n, n)$,

(3) $\bigcap_{n=1}^{\infty} \left(-3 - \frac{1}{n}, \frac{1}{n}\right)$,

(4) $\bigcup_{n=1}^{\infty} \left[-1 + \frac{1}{n}, 1 - \frac{1}{n}\right]$,

[Hint: In each case, it may be valuable to 'draw" the situation.]

Exercise 2: Let $A = \{a_1, ..., a_m\}$ be a set composed of m elements, and $B = \{b_1, ..., b_n\}$ be a set composed of n elements.

(1) Show that there exists an injective function $f : A \to B$ *if and only if* $m \leq n$.
(2) Show that there exists a surjective function $f : A \to B$ *if and only if* $m \geq n$.
(3) State and prove a *necessary and sufficient* condition (i.e. involving an '*if and only if*') for the existence of a bijective function $f : A \to B$.

Exercise 3: *(Modified version of Ex.* 18 *p.* 28 *in [Gaughan]).*

Let A and B be two sets and $f : A \to B$ be a one-to-one function;

(1) Recall the definition of the image set $Im(f) \subset B$.
(2) Show that the function $f : A \to Im(f)$ is bijective.
(3) We now suppose in addition that $Im(f) = B$, that is, $f : A \to B$ is bijective. Recall that the *inverse function* $f^{-1} : B \to A$ to f is defined by:

$$\forall b \in B,\ f^{-1}(b) = \text{the unique } a \in A \text{ such that } f(a) = b.$$

Show that:

$$\forall a \in A,\ f^{-1} \circ f(a) = a \text{ and } \forall b \in B,\ f \circ f^{-1}(b) = b.$$

Exercise 4: *(Modified version of Ex.* 14 *p.* 28 *in [Gaughan]).*

Often, when the domain of a function is not explicitely specified, it is assumed to be the set of all real numbers x for which the formula for $f(x)$ makes sense.

(1) What is the domain A of the function $f(x) = \frac{x}{x+3}$?
(2) What is the image B of this function?
(3) Show that $f : A \to B$ is bijective.
(4) Calculate the inverse function $f^{-1} : B \to A$ of f.

Exercise 5: *(Reprinted from Ex. 20 p. 28 in [Gaughan]).*
Prove that, for all $n \in \mathbb{N}^*$, $1 + 3 + 5 + ... + (2n - 1) = n^2$.

Exercise 6: Prove that, for all $n \in \mathbb{N}$, for all $x \neq 1$, one has:

$$1 + x + ... + x^n = \frac{1 - x^{n+1}}{1 - x}.$$