The Nash-Kuiper process for curves

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An isometric immersion of a Riemannian manifold into an Euclidean space is a C^1 map $f : (M^m, g) \longrightarrow \mathbb{E}^q = (\mathbb{R}^q, \langle ., . \rangle)$ such that $f^*\langle ., . \rangle = g$. Such a map preserves the length of curves that is:

$$Length(f \circ \gamma) = Length(\gamma)$$

for every rectifiable curve $\gamma : [a, b] \longrightarrow M^m$. In a local coordinate system $x = (x_1, ..., x_m)$ the isometric condition gives rise to a system of $s_m = \frac{m(m+1)}{2}$ equations

$$1 \leq i \leq j \leq n, \quad \left\langle \frac{\partial f}{\partial x_i}(x), \frac{\partial f}{\partial x_j}(x) \right\rangle = g_x \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right).$$

Thus generically, isometric maps are expected to exist –at least locally– if the target space has dimension greater than s_m. In 1926, Janet [10] proved that any analytic Riemannian surface (M^2, g) admit local isometric analytic immersions in $\mathbb{E}^{s_2}$. Shortly after, this result was generalized by Cartan [5] for analytic Riemannian manifold of dimension m: local isometric analytic maps do exist if the dimension of the target Euclidean space is at least s_m. Thirty years after the result of Janet, J. Nash showed in an outstanding article [15] that every C^k Riemannian manifold $(3 \leq k \leq +\infty)$ can be mapped C^k isometrically into an Euclidean space $\mathbb{E}^q$ with $q = 3s_m + 4m$ if M^m is compact and $q = (m+1)(3s_m+4m)$ if not. This result was then improved by Gromov [12] and Günther [9] who proved that $q = \max\{s_m+2m, s_m+m+5\}$ is enough for the compact case.

Another amazing result of J. Nash is the discovery that, in a C^1 setting, the barrier formed by the Janet dimension can be completely destroyed: in the compact case, if a Riemannian manifold admits an immersion into some $\mathbb{E}^q$, $q \geq m + 1$, then it admits a C^1 isometric immersion into the same Euclidean space (Nash [14] proved the case $q \geq m + 2$ and Kuiper [13] the case

$q = m + 1$). As a consequence every compact riemannian surface admits a C^1 isometric immersion in $\mathbb{E}^3$ but in general, for obvious curvature reasons, the immersion can not be enhanced to be C^2.

Beyond the breaking of the dimensional barrier, there is another phenomenon which is utterly baffling in the Nash-Kuiper result: not only C^1 isometric maps do exist but they are plentiful! In fact, there is a C^1 isometric map near every strictly short map. A map $f_0 : (M^m, g) \longrightarrow \mathbb{E}^q$ is called *strictly short* if it strictly shortens distances, that is, if the difference $g - f_0^*\langle ., . \rangle$ is a metric. The Nash-Kuiper approach reveals that if f_0 is a strictly short embedding, then for every $\epsilon > 0$ there exists a C^1 isometric embedding $f : (M^m, g) \longrightarrow \mathbb{E}^q$ such that

$$\|f - f_0\|_{C^0} \leq \epsilon$$

where $\|.\|_{C^0}$ denotes the supremum norm over M^m (this manifold is assumed to be compact for the simplicity of the presentation). For instance, for every $\epsilon > 0$, there is a C^1 isometric embedding of the unit sphere inside a ball of radius ϵ.

Recently [4], we have converted the Nash-Kuiper proof into an algorithm, using the Gromov convex integration theory ([12], [16], [7]). We have implemented this algorithm and produced numerical pictures of a C^1 isometric embedding f_∞ of the square flat torus $\mathbb{E}^2/\mathbb{Z}^2$ inside $\mathbb{E}^3$ that is C^0 close to a strictly short embedding f_0 of $\mathbb{E}^2/\mathbb{Z}^2$ as a torus of revolution. Our algorithm generates a sequence of maps

$$f_0, \quad f_{1,1}, f_{1,2}, f_{1,3}, \quad f_{2,1}, f_{2,2}, f_{2,3}, \quad \ldots$$

defined recursively that C^1 converges toward f_∞. The geometry of the limit map consists merely of the behavior of its tangent planes or, equivalently, of the properties of its Gauss map $\mathbf{n}_\infty : \mathbb{E}^2/\mathbb{Z} \longrightarrow \mathbb{S}^2 \subset \mathbb{E}^3$. From the algorithm, one can extract a formal expression of that Gauss map as an infinite product of *corrugation matrices* applied to the initial Gauss map of f_0. One major obstacle to the understanding of $\mathbf{n}_\infty$ lies in the inherent complexity of the coefficients of these corrugation matrices. The main theorem of [4] (the Corrugation Theorem) describes their asymptotic behaviour.

In this article, we propose to study the normal map of isometric maps resulting from a convex integration process in the simpler situation of isometric immersions of the circle $\mathbb{E}/\mathbb{Z}$ into $\mathbb{E}^2$. In this case, the isometric problem in itself is totally trivial but the way the Nash-Kuiper process solves it, produces a sequence of curves

$$f_0, f_1, f_2, \ldots$$

whose limit f_∞ has a non trivial geometry. Of course, in that one dimensional setting, some of the difficulties inherent to the dimension two vanish. In particular, if the initial curve $f_0 : \mathbb{E}/\mathbb{Z} \longrightarrow \mathbb{E}^2 \simeq \mathbb{C}$ is parametrized with constant speed and is radially symmetric (see the definition below) all computations can be completely carried out and lead to an explicit formula for the normal map $\mathbf{n}_\infty$ of the limit curve f_∞.

Theorem 1.– *Let $\mathbf{n}_k$ be the normal map of f_k. We have*

$$\forall x \in \mathbb{E}/\mathbb{Z}, \quad \mathbf{n}_k(x) = e^{i\alpha_k \cos(2\pi N_k x)} \mathbf{n}_{k-1}(x)$$

where $\alpha_k \in]0, \frac{\pi}{2}[$ is the amplitude of the loop used in the convex integration to build f_{k-1} from f_k and $N_k \in 2\mathbb{N}^*$ is the number of corrugations of f_k (precise definitions below). In particular, the normal map $\mathbf{n}_\infty$ of f_∞ has the following expression

$$\forall x \in \mathbb{E}/\mathbb{Z}, \quad \mathbf{n}_\infty(x) = \left(\prod_{k=1}^{+\infty} e^{i\alpha_k \cos(2\pi N_k x)} \right) \mathbf{n}_0(x).$$

The above expression of the normal map $\mathbf{n}_\infty$ is reminiscent of a *Riesz product*, that is a product of the form

$$h(x) = \prod_{k=1}^{+\infty} (1 + \alpha_k \cos(2\pi N_k x)).$$

It is a fact that an exponential growth of N_k, known as Hadamard's lacunary condition, results in a fractional Hausdorff dimension of the Riesz measure[1] $\mu := h(x)\mathrm{d}x$ [11].

The normal map $\mathbf{n}_\infty$ can be thought of as a 1-periodic map from $\mathbb{R}$ to $\mathbb{C}$. In §3 we perform its Fourier series expansion. Its spectrum, whose structure is very similar to the spectrum of a Riesz product, is obtained as a limit of an

[1] Let $dim_{\sup}\mu$ (resp. $dim_{\inf}\mu$) denotes the supremum (resp. the infimum) of the Hausdorff dimension of the Borel sets of positive μ-measure. If $d = dim_{\sup}\mu = dim_{\inf}\mu$ then the measure μ is said to have Hausdorff dimension d.

iterative process starting with the spectrum of the initial map $\mathbf{n}_0$. Precisely, let

$$\forall x \in \mathbb{E}/\mathbb{Z}, \quad \mathbf{n}_k(x) = \sum_{p\in\mathbb{Z}} a_p(k) e^{2i\pi px}$$

denotes the Fourier series expansion of the normal map $\mathbf{n}_k$. We derive from the above theorem the following inductive formula (*cf.* Lemma 3):

Fourier series expansion of $\mathbf{n}_k$.– *We have*

$$\forall p \in \mathbb{Z}, \quad a_p(k) = \sum_{n\in\mathbb{Z}} u_n(k) a_{p-nN_k}(k-1)$$

where $u_n(k) = i^n J_n(\alpha_k)$.

In the above formula, J_n denotes the Bessel function of order n (see [1] or [17]):

$$\alpha \longmapsto J_n(\alpha) = \frac{1}{\pi}\int_0^{\pi} \cos(nu - \alpha \sin u)\mathrm{d}u.$$

The Fourier expansion of n_k gives the key to understand the construction of the spectrum $(a_p(k))_{p\in\mathbb{Z}}$ from the spectrum $(a_p(k-1))_{p\in\mathbb{Z}}$. The k-th spectrum is obtained by collecting an infinite number of shifts of the previous spectrum. The n-th shift is of amplitude nN_k and weighted by $u_n(k) = i^n J_n(\alpha_k)$. Since

$$|J_n(\alpha_k)| \downarrow 0$$

the weight is decreasing with n (see the figure of §3).

In the Nash-Kuiper process there is a infinite number of degrees of freedom in the construction of the sequence $(f_k)_{k\in\mathbb{N}}$. In particular, given any sequence of positive numbers $(\delta_k)_{k\in\mathbb{N}}$ increasing toward 1, the process produces a sequence such that

$$\|f_k' - f_{k-1}'\|_{C^0} \leq C^{te}\sqrt{\delta_k - \delta_{k-1}}.$$

Thus, if

$$\sum \sqrt{\delta_k - \delta_{k-1}} < +\infty$$

the sequence $(f_k)_{k\in\mathbb{N}}$ is C^1 converging toward a C^1 limit f_∞. Moreover if

$$\sum \sqrt{\delta_k - \delta_{k-1}} N_k < +\infty$$

then f_∞ is C^2 (see Proposition 5). Regarding the intermediary regularities, we prove the following:

Theorem 2.– *Assume that*

$$\sum \sqrt{\delta_k - \delta_{k-1}} < +\infty \quad \text{and} \quad \sum \sqrt{\delta_k - \delta_{k-1}} N_k = +\infty.$$

Let $0 < \eta < 1$ *and* $S_k := \sum_{l=1}^{k} \sqrt{\delta_l - \delta_{l-1}} N_l$. *If*

$$\sum (\delta_k - \delta_{k-1})^{\frac{1-\eta}{2}} S_k^\eta < +\infty$$

then f_∞ *is* $C^{1,\eta}$.

In the simplified one dimensional approach followed in this article, the sequence $(N_k)_{k\in\mathbb{N}}$ can be chosen freely. This is no longer possible in the general case: some constraints appear that force the N_ks to be increasing. The control of the growth of the N_ks is then the key to understand the $C^{1,\eta}$ regularity of the limit map. In the original proof of Nash, the chosen sequence for δ_k was $1 - 2^{-(k+1)}$. For such a choice, the numerical result we have obtained for the square flat torus seems to suggest that the sequence $(N_k)_{k\in\mathbb{N}}$ is exponentially growing (see also the theoretical arguments of [6]). This gives the motivation for the following corollary.

Corollary 3.– *Let* $0 < \gamma < 1$ *and* $\delta_k := 1 - e^{-\gamma(k+1)}$. *If there exists* $\beta > 0$ *such that*

$$\forall k \in \mathbb{N}, \qquad N_k \leq N_0 e^{\beta k}$$

then f_∞ *is* $C^{1,\eta}$ *for any* $\eta > 0$ *such that*

$$\eta < \frac{\gamma}{2\beta}.$$

The question of the $C^{1,\eta}$ regularity of isometric maps resulting from the Nash-Kuiper process is addressed in [2], [3] and [6]. The optimal $C^{1,\eta}$ regularity of an isometric immersion of a Riemannian surface in $\mathbb{E}^3$ is still an open question.

1 The convex integration process for curves

The convex integration process.– Let $f_0 : [0,1] \to \mathbb{R}^2$ be a C^∞ map and let

$$\begin{array}{rccc} h: & [0,1] & \longrightarrow & C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{C}) \\ & x & \longmapsto & h(x,.) \end{array}$$

be a C^∞ family of loops such that

$$\forall x \in [0,1], \quad \int_0^1 h(x,s)\mathrm{d}s = f_0'(x).$$

Let $N \in \mathbb{N}^*$ the any natural number. We define a new C^∞ map $f : [0,1] \longrightarrow \mathbb{R}^2$ by the formula

$$\forall x \in [0,1], \quad f(x) := f_0(0) + \int_0^x h(s, \{Ns\})\mathrm{d}s$$

where $\{Ns\}$ denotes the fractional part of Ns. We call such a formula giving a new map f from the data of f_0 and h a *convex integration.* We sometimes write

$$f := IC(f_0, h, N).$$

The new map f has a derivative whose image obviously lies inside the image of h since

$$\forall x \in [0,1], \quad f'(x) = h(x, \{Nx\}).$$

Moreover, f remains C^0 close to f_0. Indeed, it can be shown that

$$\|f - f_0\|_{C^0} = O\left(\frac{1}{N}\right)$$

(see [4] for instance).

Curves with given speeds.– Let $f_0 : [0,1] \longrightarrow \mathbb{E}^2 \simeq \mathbb{C}$ be a regular curve ($\forall x \in [0,1], \ \ f_0'(x) \neq 0$) and let $r : [0,1] \longrightarrow \mathbb{R}_+^*$ be any C^∞ map such that

$$\forall x \in [0,1], \quad r(x) > \|f_0'(x)\|.$$

Let h defined by

$$h(x,s) := r(x)\left(\cos(\alpha(x)\cos 2\pi s)\mathbf{t}_0(x) + \sin(\alpha(x)\cos 2\pi s)\mathbf{n}_0(x)\right)$$

with $\mathbf{t}_0 := \frac{f_0'}{\|f_0'\|}$, $\mathbf{n}_0 := i\mathbf{t}_0$ and $\alpha(x) \in \,]0, \kappa[$ is such that

$$r(x)J_0(\alpha(x)) = \|f_0'(x)\|$$

where J_0 denotes the Bessel function of the first kind and of order 0 and $\kappa \simeq 2.4$ denotes the first zero of J_0. Since the Bessel function J_0 is decreasing on the interval $[0,\kappa]$ and $J_0(0)=1$, there is a unique $\alpha(x)$ that solves the above implicite equation. Note that

$$\int_0^1 h(x,s)\mathrm{d}s = r(x)J_0(\alpha(x))\mathbf{t}_0(x)$$

therefore the above implicit condition on $\alpha(x)$ implies that the average of $h(x,.)$ is $f_0'(x)$. The map f obtained by convex integration from f_0 and h has speed $\|f'\|$ equal to the given function r and is arbitrarily C^0 close to f_0.

Closed curves with given speeds.– If f_0 is defined over $\mathbb{E}/\mathbb{Z}$ rather than $[0,1]$ the curve f obtained from f_0 and h by convex integration is not closed in general. This defect can be easily corrected by the following modification of the convex integration formula:

$$\forall x\in[0,1],\quad f(x) := f_0(0) + \int_0^x h(s,\{Ns\})\mathrm{d}s - x\int_0^1 h(s,\{Ns\})\mathrm{d}s.$$

For short we write $f := \widetilde{IC}(f_0,h,N)$. The C^0 closeness implies that

$$\left|\int_0^1 h(x,s)\mathrm{d}s\right| = O\left(\frac{1}{N}\right)$$

so that the correction can be made arbitrarily small. We still have $\|f - f_0\|_{C^0} = O\left(\frac{1}{N}\right)$ but now $\|f'\|$ is only approximately equal to $r(x)$, precisely

$$\forall x\in\mathbb{E}/\mathbb{Z},\quad \|f'(x)\| = \left\|h(x,\{Nx\}) - \int_0^1 h(s,\{Ns\})\mathrm{d}s\right\|$$

and therefore, for all $x\in\mathbb{E}/\mathbb{Z}$, we have $|\|f'(x)\| - r(x)| = O\left(\frac{1}{N}\right)$.

Nash and Kuiper process.– In the spirit of the Nash and Kuiper proof, the way to obtain a map $f:\mathbb{E}/\mathbb{Z}\longrightarrow\mathbb{E}^2\simeq\mathbb{C}$ with speed the given function r is to produce a sequence of closed curves $(f_k)_{k\in\mathbb{N}^*}$ by iteratively applying the modifying convex integration formula so that to reduce step by step the isometric default $r - \|f_0'\|$.

Let $(\delta_k)_{k\in\mathbb{N}^*}$ be a sequence of increasing positive number converging toward 1, we set

$$\forall k\in\mathbb{N}^*, \forall x\in\mathbb{E}/\mathbb{Z},\quad r_k^2(x) := \|f_0'(x)\|^2 + \delta_k\left(r^2(x) - \|f_0'(x)\|^2\right).$$

Note that for every $x \in \mathbb{E}/\mathbb{Z}$, the sequence $r_k(x)$ is increasing toward $r(x)$. We define f_k to be $\widetilde{IC}(f_{k-1}, h_k, N_k)$ with

$$h_k(x,s) := r_k(x) e^{i\alpha_k(x)\cos 2\pi s} \mathbf{t}_{k-1}(x)$$

where $\alpha_k(x) = J_0^{-1}\left(\frac{\|f'_{k-1}(x)\|}{r_k(x)}\right)$ and $\mathbf{t}_{k-1}$ is the normalized derivative of f_{k-1}. Each f_k has a speed which is approximately r_k:

$$\big|\|f'_k(x)\| - r_k(x)\big| = O\left(\frac{1}{N_k}\right).$$

Since the sequence $r_k(x)$ is strictly increasing for every $x \in \mathbb{E}/\mathbb{Z}$, the number N_k can be chosen large enough such that

$$\forall x \in \mathbb{E}/\mathbb{Z}, \quad r_{k+1}(x) > \|f_k(x)\|.$$

This is crucial to define f_{k+1} as $\widetilde{IC}(f_k, h_{k+1}, N_{k+1})$. If the sequence $(\delta_k)_{k\in\mathbb{N}^*}$ is chosen so that

$$\sum \sqrt{\delta_k - \delta_{k-1}} < +\infty$$

and if $(N_k)_{k\in\mathbb{N}^*}$ is rapidly diverging then the sequence $f_k := \widetilde{IC}(f_{k-1}, h_k, N_k)$ is C^1 converging toward a C^1 limit f_∞ with speed $\|f'_\infty\| = r$. This is proven further in the text in the particular case of closed curves with constant speed. The general case, slightly more technical in nature, is left to the reader.

Closed curves with constant speed.– From now on, in order to get the most pleasant computations we consider the simplified case where $r \equiv 1$ and $f_0 : \mathbb{E}/\mathbb{Z} \to \mathbb{E}^2$ is a C^∞ map such that:

- $(Cond\ 1)$ it is of constant speed $r_0 := \|f'_0\| < 1$
- $(Cond\ 2)$ it is radially symmetric, that is: $f'_0(x + \frac{1}{2}) = -f'_0(x)$.

In all what follows, we will also assume that the Nash-Kuiper sequence of C^∞ maps derived from f_0:

$$f_k := \widetilde{IC}(f_{k-1}, h_k, N_k), \quad k \in \mathbb{N}^*$$

is such that $h_k(x,s) = r_k e^{i\alpha_k(x)\cos 2\pi s} \mathbf{t}_{k-1}(x)$ and $N_k \in 2\mathbb{N}^*$. Note that $(Cond\ 1)$ implies that every function $r_k = \sqrt{r_0^2 + \delta_k(1 - r_0^2)}$ is constant.

Proposition 1.– *For every* $k \in \mathbb{N}^*$*,* f_k *is of constant speed* r_k *and radially symmetric. In particular,*

$$f_k = IC(f_{k-1}, h_k, N_k).$$

The functions α_k *are also constant and equal to* $J_0^{-1}\left(\frac{r_{k-1}}{r_k}\right)$.

Proof.– By induction. Assume that f_{k-1} satisfies ($Cond$ 1) and ($Cond$ 2). In particular f_{k-1} is of constant speed r_{k-1} and thus the function $\alpha_k = J_0^{-1}\left(\frac{r_{k-1}}{r_k}\right)$ is constant. Since $N_k \in 2\mathbb{N}^*$, we have

$$h_k(x + \frac{1}{2}, \{N_k(x + \frac{1}{2})\}) = -h_k(x, \{N_k x\})$$

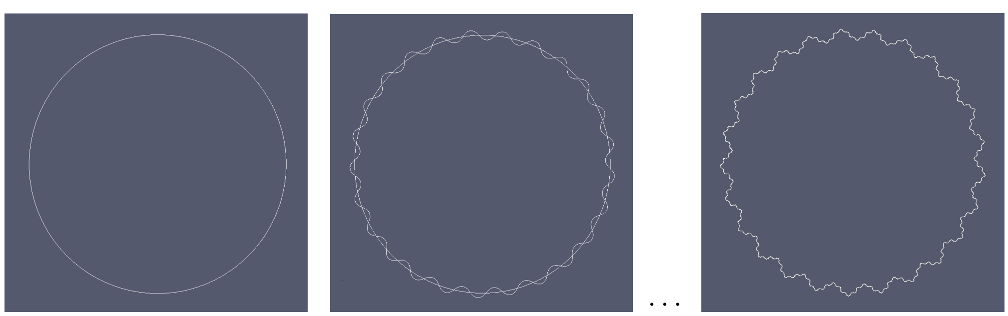
and consequently

$$\int_0^1 h_k(s, \{Ns\})\mathrm{d}s = 0.$$

It ensues that

$$IC(f_{k-1}, h_k, N_k) = \widetilde{IC}(f_{k-1}, h_k, N_k)$$

and therefore f_k is of constant speed $\|f_k'(x)\| = \|h_k(x, \{Nx\})\| = r_k$. It is also radially symmetric since $f_k(x) = h(x, \{N_k x\})$. $\square$.



The convex integration process applied to circle (left f_0*, center* f_1*, right* f_∞*)*

2 C^1 convergence

It turns out that the sequence $(\delta_k)_{k\in\mathbb{N}}$ mainly determines the sequence $(\alpha_k)_{k\in\mathbb{N}}$.

Lemma 1 (Amplitude Lemma).– *We have*

$$\alpha_k \sim \sqrt{2(1-r_0^2)}\sqrt{\delta_k-\delta_{k-1}}$$

where $\sim$ *denotes the equivalence of sequences. We also have*

$$\alpha_k \leq \frac{1}{r_0}\sqrt{2(1-r_0^2)}\sqrt{\delta_k-\delta_{k-1}}.$$

Proof.– By definition $\alpha_k = J_0^{-1}(\frac{r_{k-1}}{r_k})$. Recall that the Taylor expansion of $J_0(\alpha)$ up to order 2 is

$$w = 1 - \frac{\alpha^2}{4} + o(\alpha^2).$$

Let $y = 1-w$ and $X = \alpha^2$, we have $y = \frac{X}{4} + o(X)$ thus $X = 4y + o(y)$ and so $X \sim 4y$. We finally get

$$\alpha \sim 2\sqrt{1-w} \quad \text{and} \quad \alpha_k \sim 2\sqrt{1-\frac{r_{k-1}}{r_k}}.$$

Since $r_0^2 + (1-r_0^2) = 1$, we have

$$r_k^2 = r_0^2 + \delta_k(1-r_0^2) = 1 + (\delta_k - 1)(1-r_0^2)$$

so

$$r_k^2 - r_{k-1}^2 = (\delta_k - \delta_{k-1})(1-r_0^2)$$

and

$$1 - \frac{r_{k-1}^2}{r_k^2} = \frac{(\delta_k-\delta_{k-1})(1-r_0^2)}{1-(1-\delta_k)(1-r_0^2)} \sim (\delta_k-\delta_{k-1})(1-r_0^2).$$

In an other hand

$$1 - \frac{r_{k-1}^2}{r_k^2} = \left(1-\frac{r_{k-1}}{r_k}\right)\left(1+\frac{r_{k-1}}{r_k}\right) \sim 2\left(1-\frac{r_{k-1}}{r_k}\right).$$

Thus

$$\left(1-\frac{r_{k-1}}{r_k}\right) \sim \frac{1}{2}(\delta_k-\delta_{k-1})(1-r_0^2).$$

and

$$\alpha_k \sim 2\sqrt{1-\frac{r_{k-1}}{r_k}} \sim \sqrt{2(1-r_0^2)}\sqrt{\delta_k-\delta_{k-1}}.$$

The Taylor expansion of J_0 up to order 4 shows that

$$w \leq 1 - \frac{\alpha^2}{4} + \frac{\alpha^4}{64} = \left(1 - \frac{\alpha^2}{8}\right)^2$$

(because it is alternating) and hence

$$\alpha_k^2 \leq 8\left(1 - \sqrt{\frac{r_{k-1}}{r_k}}\right).$$

Thus

$$\begin{aligned} \alpha_k^2 &\leq \frac{8}{\sqrt{r_k}}\left(\sqrt{r_k} - \sqrt{r_{k-1}}\right) \\ &\leq \frac{8}{\sqrt{r_k}(\sqrt{r_k} + \sqrt{r_{k-1}})}\left(r_k - r_{k-1}\right) \\ &\leq \frac{8}{\sqrt{r_k}(\sqrt{r_k} + \sqrt{r_{k-1}})(r_k + r_{k-1})}\left(r_k^2 - r_{k-1}^2\right) \\ &\leq \frac{2}{r_0^2}\left(r_k^2 - r_{k-1}^2\right) \end{aligned}$$

since $r_0 < r_{k-1} < r_k$. We deduce

$$\alpha_k \leq \frac{1}{r_0}\sqrt{2(r_k^2 - r_{k-1}^2)} = \frac{1}{r_0}\sqrt{2(1 - r_0^2)}\sqrt{\delta_k - \delta_{k-1}}.$$

□

Let $(A_k)_{k \in N^*}$ be the sequence of functions defined by

$$\forall x \in \mathbb{E}/\mathbb{Z}, \quad A_k(x) := \sum_{l=1}^{k} \alpha_l \cos(2\pi N_l x).$$

Lemma 2.– *For every $x \in \mathbb{E}/\mathbb{Z}$, we have:*

$$f_k'(x) = e^{iA_k(x)}\frac{r_k}{r_0}f_0'(x).$$

Proof.– Let $\mathbf{n}_{k-1} := i\mathbf{t}_{k-1}$. From

$$\begin{aligned} f_k'(x) &= r_k(\cos(\alpha_k \cos(2\pi N_k x))\mathbf{t}_{k-1}(x) + \sin(\alpha_k \cos(2\pi N_k x))\mathbf{n}_{k-1}(x)) \\ &= r_k e^{i\alpha_k \cos(2\pi N_k x)}\frac{1}{r_{k-1}}f_{k-1}'(x) \end{aligned}$$

we deduce by induction : $f_k'(x) = e^{iA_k(x)}\frac{r_k}{r_0}f_0'(x)$. □

Proposition 2.– *If* $\sum \sqrt{\delta_k - \delta_{k-1}} < +\infty$ *then*

i) the sequence $(A_k)_{k\in\mathbb{N}}$ *is normally converging and* $A_\infty := \lim_{k\to+\infty} A_k$ *is continuous.*

ii) the sequence $(f_k)_{k\in\mathbb{N}^*}$ *is* C^1 *converging toward* $f_\infty := \lim_{k\to+\infty} f_k$ *and*

$$\forall\, x \in \mathbb{R}/\mathbb{Z}, \qquad f'_\infty(x) = e^{iA_\infty(x)} \frac{1}{r_0} f'_0(x).$$

Proof.– From the Amplitude Lemma we deduce that

$$\sum \alpha_k < +\infty$$

thus the sequence $(A_k)_{k\in\mathbb{N}}$ is normally converging and

$$A_\infty := \lim_{k\to+\infty} A_k$$

is continuous. Moreover, from the relation

$$f'_k(x) = e^{iA_k(x)} \frac{r_k}{r_0} f'_0(x)$$

we also deduce that $\left(f'_k\right)_{k\in\mathbb{N}}$ is normally converging toward

$$e^{iA_\infty(x)} \frac{1}{r_0} f'_0(x).$$

Since $(f_k(0))_{k\in\mathbb{N}}$ obviously converges, we obtain that the sequence $(f_k)_{k\in\mathbb{N}}$ is C^1 converging toward $f_\infty := \lim_{k\to+\infty} f_k$. $\square$

Corollary 1.– *Let* $\gamma > 0$ *and* $\delta_k := 1 - e^{-\gamma(k+1)}$. *Then sequence* $(\delta_k)_{k\in\mathbb{N}^*}$ *is increasing toward 1 and* $\sqrt{\delta_k - \delta_{k-1}} \sim \sqrt{\delta_0} e^{-\frac{\gamma}{2}k}$. *In particular,* f_∞ *is* C^1.

3 The normal map

From now on, we assume

$$\sum \sqrt{\delta_k - \delta_{k-1}} < +\infty$$

so that the sequence $(f_k)_{k\in\mathbb{N}}$ is C^1 converging toward its limit f_∞. The following theorem is a straightforward consequence of the results of the preceding section:

Theorem 1.– *Let* $\mathbf{n}_k$ *be the normal map of* f_k*. We have*

$$\forall x \in \mathbb{E}/\mathbb{Z}, \quad \mathbf{n}_k(x) = e^{i\alpha_k \cos(2\pi N_k x)} \mathbf{n}_{k-1}(x)$$

In particular, the normal map $\mathbf{n}_\infty$ *of* f_∞ *has the following expression*

$$\forall x \in \mathbb{E}/\mathbb{Z}, \quad \mathbf{n}_\infty(x) = e^{iA_\infty(x)} \mathbf{n}_0(x).$$

We deduce from this theorem the following result about Fourier expansion of $\mathbf{n}_k$.

Lemma 3 (Fourier expansion of $\mathbf{n}_k$).– *For all* $k \in \mathbb{N}$ *we denote by*

$$\forall x \in \mathbb{E}/\mathbb{Z}, \quad \mathbf{n}_k(x) = \sum_{p\in\mathbb{Z}} a_p(k) e^{2i\pi p x}$$

the Fourier expansion of $\mathbf{n}_k$*. We have*

$$\forall p \in \mathbb{Z}, \quad a_p(k) = \sum_{n\in\mathbb{Z}} u_n(k) a_{p-nN_k}(k-1)$$

where $u_n(k) = i^n J_n(\alpha_k)$ *(*J_n *denotes the Bessel function of order* n*).*

Proof.– From the Jacobi-Anger identity

$$e^{iz\cos\theta} = \sum_{n=-\infty}^{+\infty} i^n J_n(z) e^{in\theta}$$

we deduce

$$e^{i\alpha_k \cos(2\pi N_k x)} = \sum_{n=-\infty}^{+\infty} i^n J_n(\alpha_k) e^{2i\pi n N_k x} = \sum_{n=-\infty}^{+\infty} u_n(k) e^{2i\pi n N_k x}.$$

Since the Fourier coefficients of a product of two fonctions are given by the discrete convolution product of their coefficients, the product

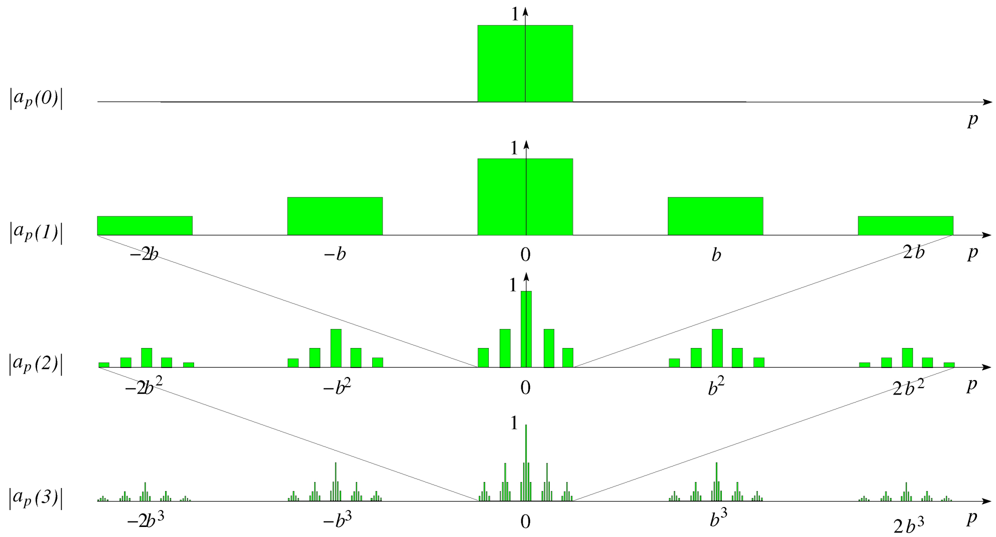
$$\mathbf{n}_k(x) = e^{i\alpha_k \cos(2\pi N_k x)} \mathbf{n}_{k-1}(x)$$

can be written

$$\begin{aligned} \mathbf{n}_k(x) &= \left(\sum_{n=-\infty}^{+\infty} u_n(k) e^{2i\pi n N_k x} \right) \left(\sum_{p=-\infty}^{+\infty} a_p(k-1) e^{2i\pi p x} \right) \\ &= \sum_{p=-\infty}^{+\infty} \left(\sum_{n=-\infty}^{+\infty} u_n(k) a_{p-nN_k}(k-1) \right) e^{2i\pi p x}. \end{aligned}$$

Therefore

$$a_p(k) = \sum_{n=-\infty}^{+\infty} u_n(k) a_{p-nN_k}(k-1). \qquad \square$$



A schematic picture of the various spectra $(a_p(k))_{p\in\mathbb{Z}}$ *with* $N_k = b^k$.

Remark.– The analogy with Riesz products suggests that the Hausdorff dimension of the graph of the normal map $\mathbf{n}_\infty$ could be fractional. Note that the relevant part of this map is the 1-periodic function

$$\mathbb{R} \ni x \longmapsto A_\infty(x) = \sum_{k=1}^{+\infty} \alpha_k \cos(2\pi N_k x) \in \mathbb{R}.$$

In the simple case where $\alpha_k = a^k$ and $N_k = b^k$ with $0 < a < 1 < b$, the map A_∞ is a Weiestrass function[2]. If $ab > 1$, it is known that its graph has a fractional Hausdorff dimension. The exact value of this dimension is still an open question. It is believed to be equal to $2 + \frac{\ln a}{\ln b}$ (see [8]).

4 $C^{1,\eta}$ regularity

Proposition 3.– *We have*

$$\|f'_k - f'_{k-1}\|_{C^0} \leq Cte_1 \sqrt{\delta_k - \delta_{k-1}}$$

[2] *Cf.* Lemma 1 and the lines above Corollary 3 in the introductory part of this article for a motivation for such choice for α_k and N_k.

with $Cte_1 = \sqrt{7(1-r_0^2)}$.

Proof.– For every point $x \in \mathbb{E}/\mathbb{Z}$, we have

$$\|f'_k - f'_{k-1}\|^2 = \|f'_k\|^2 + \|f'_{k-1}\|^2 - 2\|f'_k\|\|f'_{k-1}\|\cos(\alpha_k \cos 2\pi N_k x)$$

since $\alpha_k \cos 2\pi N_k x$ is the angle between $f'_k(x)$ and $f'_{k-1}(x)$. An upper bound for this angle is $\alpha_k = J_0^{-1}(w)$ where $w = r_{k-1}/r_k \in]0,1[$ since

$$r_k = \|f'_k(x)\| \quad \text{and} \quad r_{k-1} = \|f'_{k-1}(x)\|.$$

Recall that from the Amplitude Lemma we have the following inequality

$$\frac{\alpha_k^2}{2} \leq 4(1-\sqrt{w}).$$

By using the upper bound α_k, we obtain

$$\begin{array}{rcl} \|f'_k - f'_{k-1}\|^2 & \leq & r_k^2 + r_{k-1}^2 - 2r_{k-1}r_k \cos\alpha_k \\ & \leq & r_k^2 - r_{k-1}^2 + 2r_{k-1}(r_{k-1} - r_k \cos\alpha_k). \end{array}$$

Since

$$\cos\alpha_k \geq 1 - \frac{\alpha_k^2}{2}$$

we have

$$\begin{array}{rcl} r_{k-1}(r_{k-1} - r_k\cos\alpha_k) & \leq & r_{k-1}^2 - r_k r_{k-1} + r_{k-1} r_k \frac{\alpha_k^2}{2} \\ & \leq & r_{k-1}^2 - r_k r_{k-1} + 4r_{k-1}r_k \left(1 - \sqrt{\frac{r_{k-1}}{r_k}}\right) \\ & \leq & r_{k-1}^2 + 3r_{k-1}r_k - 4r_{k-1}\sqrt{r_k r_{k-1}} \\ & \leq & r_{k-1}^2 + 3r_k^2 - 4r_{k-1}\sqrt{r_{k-1}^2} \quad (\text{ since } \ r_{k-1} < r_k) \\ & \leq & 3(r_k^2 - r_{k-1}^2). \end{array}$$

Therefore

$$\|f'_k - f'_{k-1}\|^2 \leq 7\left(\|f'_k\|^2 - \|f'_{k-1}\|^2\right).$$

Now

$$\begin{array}{rcl} \|f'_k\|^2 - \|f'_{k-1}\|^2 & = & r_k^2 - r_{k-1}^2 \\ & = & (\delta_k - \delta_{k-1})(1 - r_0^2). \end{array}$$

Finally

$$\|f'_k - f'_{k-1}\|_{C^0} \leq Cte_1\sqrt{\delta_k - \delta_{k-1}}$$

with $Cte_1 = \sqrt{7(1-r_0^2)}$. □

For every $k \in \mathbb{N}$, we denote by $M_k(g)$ the supremum over $\mathbb{E}/\mathbb{Z}$ of the k-th derivative $g^{(k)}$ of $g : \mathbb{E}/\mathbb{Z} \longrightarrow \mathbb{C}$ (if $k = 0$, it is understood that $g^{(0)} = g$) and we define $\|g\|_{C^k}$ to be the sum $M_0(g) + ... + M_k(g)$.

Corollary 2.– *We have*

$$\|f_k - f_{k-1}\|_{C^1} \leq 2Cte_1\sqrt{\delta_k - \delta_{k-1}}$$

with $Cte_1 = \sqrt{7(1-r_0^2)}$.

Proof.– From the theorem we deduce by a mere integration

$$\|f_k - f_{k-1}\|_{C^0} \leq Cte_1\sqrt{\delta_k - \delta_{k-1}}$$

thus the result since

$$\|f_k - f_{k-1}\|_{C^1} = \|f_k - f_{k-1}\|_{C^0} + M_1(f_k - f_{k-1}).$$

□

Proposition 4.– *For every* $x \in \mathbb{E}/\mathbb{Z}$, *we have*

$$f_k''(x) = \left(-2\pi\alpha_k N_k \sin 2\pi N_k x + r_{k-1} scal_{k-1}(x)\right) i r_k e^{i\alpha_k \cos 2\pi N_k x}\mathbf{t}_{k-1}(x)$$

where $scal_k$ *denotes the signed curvature of* f_k. *Moreover*

$$r_k scal_k(x) = r_0 scal_0(x) - 2\pi \sum_{l=1}^{k} \alpha_l N_l \sin(2\pi N_l x).$$

Proof.– We have

$$\begin{aligned}
f_k''(x) &= \frac{\partial}{\partial x}\left(r_k e^{i\alpha_k \cos 2\pi N_k x}\mathbf{t}_{k-1}(x)\right) \\
&= \frac{\partial}{\partial x}\left(r_k(\cos(\alpha_k \cos 2\pi N_k x)\mathbf{t}_{k-1}(x) + \sin(\alpha_k \cos 2\pi N_k x)\mathbf{n}_{k-1}(x)\right) \\
&= \frac{r_k}{r_{k-1}}\frac{\partial}{\partial x}\left(\cos(\alpha_k \cos 2\pi N_k x) f_{k-1}'(x) + \sin(\alpha_k \cos 2\pi N_k x) i f_{k-1}'(x)\right) \\
&= -2i\pi\alpha_k N_k \sin(2\pi N_k x) r_k e^{i\alpha_k \cos 2\pi N_k x}\mathbf{t}_{k-1}(x) \\
&\quad + \frac{r_k}{r_{k-1}}\left(\cos(\alpha_k \cos 2\pi N_k x) f_{k-1}''(x) + \sin(\alpha_k \cos 2\pi N_k x) i f_{k-1}''(x)\right)
\end{aligned}$$

Since f_{k-1} is of constant speed r_{k-1} we have

$$f_{k-1}''(x) = r_{k-1} scal_{k-1}(x) i f_{k-1}'(x)$$

therefore

$$\begin{array}{rcl} f_k''(x) & = & -2i\pi\alpha_k N_k \sin(2\pi N_k x) r_k e^{i\alpha_k \cos 2\pi N_k x}\mathbf{t}_{k-1}(x) \\ & & +r_k r_{k-1} scal_{k-1}(x) i e^{i\alpha_k \cos 2\pi N_k x}\mathbf{t}_{k-1}(x) \end{array}.$$

Finally,

$$f_k''(x) = \left(-2\pi\alpha_k N_k \sin 2\pi N_k x + r_{k-1} scal_{k-1}(x)\right) i r_k e^{i\alpha_k \cos 2\pi N_k x}\mathbf{t}_{k-1}(x).$$

Because f_k is of constant arc length we also have

$$f_k''(x) = r_k scal_k(x) i f_k'(x) = r_k scal_k(x) i r_k e^{i\alpha_k \cos 2\pi N_k x}\mathbf{t}_{k-1}(x).$$

From this we deduce

$$r_k scal_k(x) = r_{k-1} scal_{k-1}(x) - 2\pi\alpha_k N_k \sin(2\pi N_k x)$$

and by induction

$$r_k scal_k(x) = r_0 scal_0(x) - 2\pi \sum_{l=1}^{k} \alpha_l N_l \sin(2\pi N_l x).$$

□

Proposition 5.– *If* $\sum_{k\in\mathbb{N}^*} \sqrt{\delta_k - \delta_{k-1}} N_k < +\infty$ *then* f_∞ *is* C^2.

Proof.– Since we already know that the sequence $(f_k)_{k\in\mathbb{N}}$ C^1 converges, it is enough to prove that $(f_k'')_{k\in\mathbb{N}}$ is a Cauchy sequence. From

$$f_k''(x) = r_k scal_k(x) i f_k'(x)$$

we deduce

$$\begin{array}{rcl} \|f_k''(x) - f_{k-1}''(x)\|_{C^0} & \leq & \|r_k scal_k(x) f_k'(x) - r_{k-1} scal_{k-1}(x) f_{k-1}'(x)\|_{C^0} \\ & \leq & \|r_{k-1} scal_{k-1}(x) f_k'(x) - r_{k-1} scal_{k-1}(x) f_{k-1}'(x)\|_{C^0} \\ & & +|r_k scal_k(x) - r_{k-1} scal_{k-1}(x)| \|f_k'(x)\|_{C^0} \\ & \leq & r_{k-1} |scal_{k-1}(x)| \|f_k'(x) - f_{k-1}'(x)\|_{C^0} \\ & & +r_k |r_k scal_k(x) - r_{k-1} scal_{k-1}(x)|. \end{array}$$

Since

$$r_k scal_k(x) = r_0 scal_0(x) - 2\pi \sum_{l=1}^{k} \alpha_l N_l \sin(2\pi N_l x)$$

we have

$$|r_k scal_k(x) - r_{k-1} scal_{k-1}(x)| \leq 2\pi\alpha_k N_k$$

and

$$r_k |scal_k(x)| \leq r_0 |scal_0(x)| + 2\pi \sum_{l\in\mathbb{N}^*} \alpha_l N_l.$$

In particular the $r_k|scal_k(x)|$ are uniformly bounded by

$$M := \|r_0 scal_0(x)\|_{C^0} + 2\pi \sum_{k\in\mathbb{N}^*} \alpha_k N_k.$$

Note that $M < +\infty$. Indeed $\alpha_k \sim \sqrt{2(1-r_0^2)}\sqrt{\delta_k - \delta_{k-1}}$ therefore

$$\sum_{k\in\mathbb{N}^*} \sqrt{\delta_k - \delta_{k-1}} N_k < +\infty \quad \Longrightarrow \quad \sum_{k\in\mathbb{N}^*} \alpha_k N_k < +\infty.$$

We deduce

$$\|f_k''(x) - f_{k-1}''(x)\|_{C^0} \leq M \|f_k'(x) - f_{k-1}'(x)\|_{C^0} + 2\pi\alpha_k N_k.$$

Let $p < q$, we thus have

$$\begin{aligned} \|f_q''(x) - f_p''(x)\|_{C^0} &\leq M \sum_{k=p}^{q} \sqrt{\delta_k - \delta_{k-1}} + 2\pi \sum_{k=p}^{q} \alpha_k N_k \\ &\leq M \sum_{k=p}^{\infty} \sqrt{\delta_k - \delta_{k-1}} + 2\pi \sum_{k=p}^{\infty} \alpha_k N_k. \end{aligned}$$

Hence $\left(f_k''\right)_{k\in\mathbb{N}}$ is a Cauchy sequence. □

Theorem 2.– *Assume that*

$$\sum \sqrt{\delta_k - \delta_{k-1}} < +\infty \quad \text{and} \quad \sum \sqrt{\delta_k - \delta_{k-1}} N_k = +\infty.$$

Let $0 < \eta < 1$ *and* $S_k := \sum_{l=1}^{k} \sqrt{\delta_l - \delta_{l-1}} N_l$*. If*

$$\sum (\delta_k - \delta_{k-1})^{\frac{1-\eta}{2}} S_k^{\eta} < +\infty$$

then f_∞ *is* $C^{1,\eta}$.

Proof.– Let $0 < \eta < 1$. We are going to use the interpolation inequality

$$\|f\|_{C^{1,\eta}} \leq C^{te} \|f\|_{C^1}^{1-\eta} \|f\|_{C^2}^{\eta}$$

to show that $(\|f_k - f_{k-1}\|_{C^{1,\eta}})_{k\in\mathbb{N}^*}$ is a Cauchy sequence. From the above sections, we have

$$\|f_k - f_{k-1}\|_{C^1} \leq 2Cte_1\sqrt{\delta_k - \delta_{k-1}}$$

and

$$\begin{aligned}
M_2(f_k - f_{k-1}) &\leq M_2(f_k) + M_2(f_{k-1}) \\
&\leq M_0(r_k\ scal_k)M_1(f_k) + M_0(r_{k-1}\ scal_{k-1})M_1(f_k) \\
&\leq M_0(scal_k) + M_0(scal_{k-1}) \\
&\leq 2M_0(scal_0) + 4\pi\sum_{l=1}^{k}\alpha_l N_l.
\end{aligned}$$

From the Amplitude Lemma we deduce

$$\begin{aligned}
M_2(f_k - f_{k-1}) &\leq 2M_0(scal_0) + \tfrac{4\pi\sqrt{2(1-r_0^2)}}{r_0}\textstyle\sum_{l=1}^{k}\sqrt{\delta_l - \delta_{l-1}}N_l \\
&\leq 2M_0(scal_0) + \tfrac{4\pi\sqrt{2(1-r_0^2)}}{r_0}S_k.
\end{aligned}$$

So

$$\|f_k - f_{k-1}\|_{C^2} \leq 2Cte_1\sqrt{\delta_k - \delta_{k-1}} + 2M_0(scal_0) + \frac{4\pi\sqrt{2(1-r_0^2)}}{r_0}S_k.$$

Since $\lim_{k\to+\infty} S_k = +\infty$, for k large enough we have

$$\|f_k - f_{k-1}\|_{C^2} \leq Cte_2 S_k.$$

for some constant Cte_2. We now have

$$\|f_k - f_{k-1}\|_{C^1}^{1-\eta}\|f_k - f_{k-1}\|_{C^2}^{\eta} \leq Cte_3(\delta_k - \delta_{k-1})^{\frac{1-\eta}{2}}S_k^{\eta}$$

with $Cte_3 = (2Cte_1)^{1-\eta}Cte_2^{\eta}$. □

Corollary 3.– *Let* $0 < \gamma < 1$ *and* $\delta_k := 1 - e^{-\gamma(k+1)}$. *If there exists* $\beta > 0$ *such that*

$$\forall k \in \mathbb{N}, \qquad N_k \leq N_0 e^{\beta k}$$

then f_∞ *is* $C^{1,\eta}$ *for any* $\eta > 0$ *such that*

$$\eta < \frac{\gamma}{2\beta}.$$

Proof.– We have

$$\delta_k - \delta_{k-1} = \delta_0 e^{-\gamma k}$$

thus

$$S_k = \sum_{l=1}^{k} \sqrt{\delta_l - \delta_{l-1}} N_l \leq \sqrt{\delta_0} N_0 \sum_{l=1}^{k} e^{(\beta - \frac{\gamma}{2})l} < \sqrt{\delta_0} N_0 e^{\beta - \frac{\gamma}{2}} \frac{1 - e^{\left(\beta - \frac{\gamma}{2}\right)(k+1)}}{1 - e^{\beta - \frac{\gamma}{2}}}$$

Suppose first that $\beta > \dfrac{\gamma}{2}$. We then have :

$$S_k \leq Cte_4 e^{(\beta - \frac{\gamma}{2})k}.$$

Finally

$$(\delta_k - \delta_{k-1})^{\frac{1-\eta}{2}} S_k^{\eta} \leq Cte_5 e^{-\gamma \frac{1-\eta}{2} k} e^{\eta(\beta - \frac{\gamma}{2})k}.$$

Now

$$-\gamma \frac{1-\eta}{2} + \eta \left(\beta - \frac{\gamma}{2} \right) < 0$$

if and only if

$$\eta < \frac{\gamma}{2\beta}.$$

Therefore, under that condition

$$\sum (\delta_k - \delta_{k-1})^{\frac{1-\eta}{2}} S_k^{\eta} < +\infty$$

hence the corollary in the case where $\beta > \dfrac{\gamma}{2}$. We left to the reader the easier case $\beta \leq \dfrac{\gamma}{2}$. □

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