The VC dimension of k-uniform random hypergraphs

B. Ycart* and J. Ratsaby**

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Addresses: * LMC, CNRS UMR 5523, BP 53, 38041 Grenoble cedex 9, FRANCE
** Ben Gurion University of the Negev, P.O.B. 653, Beer-Sheva 84105, ISRAEL

E-mail addresses: * bernard.ycart@imag.fr
** ratsaby@bgu.ac.il

Corresponding author : B. Ycart

Abstract

A set of vertices is shattered in a hypergraph if any of its subsets is obtained as the intersection of an edge with the set. The VC dimension is the size of the largest shattered subset. Under the binomial model of k-uniform random hypergraphs, the threshold function for the VC dimension to be larger than a given integer is obtained. The same is done for the testing dimension, which is the largest integer d such that all sets of cardinality d are shattered.

Keywords: random hypergraph, Vapnik-Chervonenkis, testing dimension, threshold

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1 Introduction

The literature on VC dimensions is quite rich as it has been instrumental in the theory of empirical processes (Pollard [9]), statistical learning theory (Vapnik [13]), combinatorial geometry (Matoušek [8]), and the Probably Approximately Correct (PAC) model (Haussler [4]). The basic definitions are the following (see Vapnik [13] for a general reference).

Definition 1.1 *Let X be a finite set, and $S = \{x_1, \ldots, x_k\}$ be a subset of X. Let F be a class of functions from X to $\{0, 1\}$. For any function f, denote by $f_{|S}$ its restriction to S: $f_{|S} = [f(x_1), \ldots, f(x_k)]$.*

1. *The class F is said to* shatter *S if $\left|\{f_{|S} : f \in F\}\right| = 2^k$.*

2. *The* Vapnik-Chervonenkis *(VC) dimension of F, denoted as $VC(F)$, is the cardinality of the largest set shattered by F.*

3. *The* testing dimension *of F, denoted as $TD(F)$, is the maximal integer d such that* all *sets of size d are shattered by F.*

In this paper, we choose to represent subsets of X, or functions from X to $\{0, 1\}$, by hyperedges $e \in E$ of a hypergraph $H = (X, E)$ with a set of vertices X. The above definitions extend to hypergraphs in the obvious way: if $H = (X, E)$ is a hypergraph then a subset $S \subseteq X$ is shattered by H if for every $A \subseteq S$ there exists a hyperedge $e \in E$ such that $e \cap S = A$. The VC dimension of H is the cardinality of the largest shattered subset of X. Note that for Anthony et al. [1], the VC dimension of a graph is relative to vertex neighborhood sets, instead of edges.

We shall only deal here with hypergraphs whose hyperedges have the same cardinality k, i.e. k-uniform hypergraphs. The size k of edges is fixed whereas the number n of vertices tends to infinity. The literature on random hypergraphs is much smaller than that of random graphs. One of the reasons, as pointed out by Karoński and Łuczak [6], p. 284, is that "in many instances, results from random graphs can be generalised to random hypergraphs with some minor difficulty". This is certainly the case for all the results used here, which concern thresholds for the containment of small subgraphs and extension properties. For that reason, and in order to avoid cumbersome repetitions of the prefix 'hyper', k-uniform hypergraphs will be referred to simply as graphs with edges of cardinality k. We shall use [5, 11] as general references on random graphs. Our model is the so called binomial model $\mathcal{G}_k(n, p_n)$ where the probability of a particular graph G having e edges is:

$$Prob(\mathcal{G}_k(n, p_n) = G) = p_n^e (1 - p_n)^{\binom{n}{k} - e} .$$

We are interested in the following two graph properties.

$$E_h = \text{"there exists a set of } h \text{ vertices which is shattered."} \quad (1)$$
$$A_h = \text{"all sets of } h \text{ vertices are shattered."} \quad (2)$$

A graph G satisfies E_h if and only if its VC dimension is no smaller than h. It satisfies A_h if and only if its testing dimension is no smaller than h. Our main results give the thresholds for E_h and A_h under the binomial model $\mathcal{G}_k(n, p_n)$.

Theorem 1.2 *Let k and h be two integers such that $1 \leq h \leq k$. Define $r_n(k, h)$ as*

$$r_n(k, h) = n^{-k+h(h-1)/(h+1)} . \quad (3)$$

In the model $\mathcal{G}_k(n, p_n)$ the threshold for E_h is $r_n(k, h)$.

Theorem 1.3 *Let k and h be two integers such that $1 \leq h < k$. Define $s_n(k, h)$ as*

$$s_n(k, h) = (\log n) n^{-k+h} . \quad (4)$$

In the model $\mathcal{G}_k(n, p_n)$ the threshold for A_h is $s_n(k, h)$.

As a consequence, it turns out that the VC and the testing dimensions are concentrated on one or two values for most reasonable assignments of p_n. The proofs use classical techniques and results from the theory of random graphs, which are recalled in Section 2. A key observation is that properties E_h and A_h can be expressed respectively as containment and extension properties, relative to some particular subgraphs, called the "shattering graphs" (Definition 3.1). They are studied in Section 3. Theorems 1.2 and 1.3 are proved in section 4.

2 Threshold functions

This section reviews some of the most basic tools of the theory of random hypergraphs: the thresholds for containment and extension properties.

Recall that a graph property is monotone if as soon as it is true for some graph, it remains true for any graph containing it as a subgraph. Our basic example of a monotone property is the shattering of sets: if a given set is shattered in some graph and edges are added to that graph, then it remains shattered in the new graph. It is well known that under the model $\mathcal{G}_k(n, p_n)$, the probability of a non trivial monotone property is an increasing function of the probability parameter p_n. The value of p_n for which it goes from almost 0 to almost 1, is the *threshold* (see Spencer [11] or Janson et al. [5] for general references). For two sequences of positive reals, (p_n), (r_n), we write $p_n \ll r_n$ to denote that p_n is negligible compared to r_n, i.e.,

$$p_n \ll r_n \iff \lim_{n\to\infty} \frac{p_n}{r_n} = 0 .$$

The fact that some sequence (r_n) is a threshold for some graph property is usually expressed by saying that below the threshold (for $p_n \ll r_n$), A is false *w.h.p.* (with high probability), and above the threshold (for $p_n \gg r_n$), A is true *w.h.p.*

If G is a graph, the containment property $Cont(G)$ is satisfied by a graph F if there exists a graph isomorphism between G and some subgraph of F (see Ch. 3 of [5]). The *density* of G, denoted by $d(G)$, is the ratio of the number of edges to the number of vertices. Its *maximal density*, denoted by $m(G)$ is the highest density of a subgraph of G. Theorem 2.1 gives the threshold for the containment property $Cont(G)$. Some of its particular cases date back to the pioneering paper of Erdős and Rényi [3] and it was later completed by Bollobás [2]. The generalization to hypergraphs was stated by Vantsyan [12].

Theorem 2.1 *Let G be a graph with maximal density $m(G)$. In the model $\mathcal{G}_k(n, p_n)$, the threshold for the containment property $Cont(G)$ is $n^{-1/m(G)}$.*

Theorem 2.1 will be used in Section 4 to obtain the threshold of E_h.

The threshold of A_h is related to extension statements (see Section 3.4 of [5] and [10]) which are defined as follows. Let $R = \{x_1, \dots, x_h\}$ be a set of vertices in a graph G, such that no edge of G is included in R. The pair (R, G) is called a *rooted graph*. A graph F satisfies the *extension property $Ext(R, G)$* if for every h-tuple R' of vertices of F, there exists a graph isomorphism from G onto a subgraph of F that maps R onto R'. The rooted density $d(R, G)$ is the ratio of the number of edges to the number of vertices, not counting vertices in R. The *maximal rooted density* $m(R, G)$ is the highest density of a subgraph of G rooted on R. The *root coefficient* $s(R, G)$ is defined as the smallest number of edges in a subgraph H of G with maximal density $m(R, G)$ such that at least one of the roots $x_1, \dots, x_h$ is not isolated in H. If no such graph exists, then $s(R, G) = \infty$.

Theorem 2.2 *Let (R, G) be a rooted graph with maximal rooted density $m(R, G)$ and root coefficient $s(R, G)$. The threshold for the extension property $Ext(R, G)$ is:*

$$(\log n)^{1/s(R,G)} n^{-1/m(R,G)} .$$

Actually, the results proved by Spencer [10] for ordinary graphs go beyond the expression of the threshold (see also Theorem 3.27 p. 74 of [5]). To the best of our knowledge, Theorem 2.2 has never been stated for k-uniform random hypergraphs. However, it cannot be considered as new, since neither its statement nor its proof based on Janson's inequalities depend on k.

Theorems 2.1 and 2.2 will be applied to particular graphs associated with shattering of subsets. These graphs are described in the next section.

3 Shattering graphs

In this section, the shattering of a given set of vertices is shown to relate to the existence of certain subgraphs whose maximal density is given in Proposition 3.2.

Recall that since all edges have k elements, only sets with at most k elements can be shattered. Let $h \leq k$, and $R = \{x_1, \ldots, x_h\}$ be a set of h vertices in some graph G. For R to be shattered, G must contain at least 2^h edges of k vertices, some of them among $\{x_1, \ldots, x_h\}$. More precisely, by Definition 1.1, G shatters R if and only if for $l = 0, \ldots, h$, and for any subset $R_l = \{x_{i_1}, \ldots, x_{i_{h-l}}\}$ of R, G has an edge whose intersection with R is R_l. Thus, for all $l = 0, \ldots, h$ there must be $\binom{h}{l}$ edges, each containing $h - l$ vertices among $\{x_1, \ldots, x_h\}$, plus $k - h + l$ others. This involves necessarily 2^h edges, and a certain number of vertices that depends on whether the edges share vertices outside R. If no vertex outside R is common to two edges, the density is minimal, because the number of vertices is maximal. That number is:

$$k + h(k - h + 1) + \cdots + \binom{h}{l}(k - h + l) + \cdots + h(k - 1) + k \; .$$

In Definition 3.1 below, one should see the last shattering graph $S_{k,h,h}$ as one of the graphs G such that $cont(G)$ implies that $\{x_1, \ldots, x_h\}$ is shattered. Among those graphs, $S_{k,h,h}$ has minimal density. For $l = 0, \ldots, h$, we denote by $e_{h,l}$ and $v_{k,h,l}$ the following integers.

$$\begin{aligned} e_{h,l} &= 1 + \cdots + \tbinom{h}{l} \\ v_{k,h,l} &= (k - h) + h(k - h + 1) + \cdots + \tbinom{h}{l}(k - h + l) \; . \end{aligned} \tag{5}$$

The *shattering graphs* are constructed as a nested sequence:

Definition 3.1 *For $1 \leq h \leq k$, we call* shattering graphs, *and denote by $(S_{k,h,l})_{0 \leq l \leq h}$, the $h + 1$ graphs defined as follows:*

1. *For $l = 1, \ldots, h$, $S_{k,h,l-1}$ is a subgraph of $S_{k,h,l}$.*

2. *$S_{k,h,l}$ has $h + v_{k,h,l}$ vertices, labelled $x_1, \ldots, x_h, y_1, \ldots, y_{v_{k,h,l}}$.*

3. *For all $\ell = 0, \ldots, l$ and for any subset $\{x_{i_1}, \ldots, x_{i_{h-\ell}}\}$ of size $h - \ell$ of $\{x_1, \ldots, x_h\}$, $S_{k,h,l}$ contains a single edge whose intersection with $\{x_1, \ldots, x_h\}$ is $\{x_{i_1}, \ldots, x_{i_{h-\ell}}\}$.*

4. *Any vertex $y_1, \ldots, y_{v_{k,h,h}}$ belongs to a unique edge.*

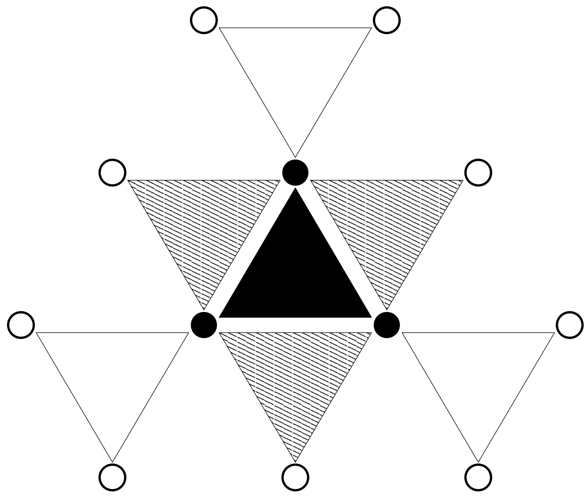

Figure 1: The 3 nested graphs $S_{3,3,0}$ (black), $S_{3,3,1}$ (black and dashed), and $S_{3,3,2}$. Vertices x_1, x_2, x_3 appear as solid dots while edges joining 3 vertices are triangles. The graph $S_{3,3,1}$ has maximal density $\frac{2}{3}$.

Figure 1 shows the shattering graphs $S_{3,3,0}$, $S_{3,3,1}$ and $S_{3,3,2}$. To understand the relation with properties E_h and A_h, it is enough at this point to remark that the containment property $Cont(S_{k,h,h})$ implies E_h, and that the extension property $Ext(R, S_{k,h,h})$ implies A_h. The thresholds for $Cont(S_{k,h,h})$ and $Ext(R, S_{k,h,h})$ depend on the maximal density of $S_{k,h,h}$ as a graph, and as a rooted graph. They are given in Proposition 3.2.

Proposition 3.2

1. *Assume $1 \leq h \leq k$. Among all subgraphs of $S_{k,h,h}$, $S_{k,h,1}$ has the maximal density,*

$$m(S_{k,h,h}) = d(S_{k,h,1}) = \frac{1+h}{k + h(k-h+1)} \; .$$

2. *Assume* $1 \leq h < k$. *Among all subgraphs of the rooted graph* $(R, S_{k,h,h})$, $(R, S_{k,h,0})$ *has the maximal rooted density,*

$$m(R, S_{k,h,h}) = d(R, S_{k,h,0}) = \frac{1}{k-h} \ .$$

Proof: To any edge of $S_{k,h,h}$, we associate its *level*, which is the smallest integer l such that the edge belongs to $S_{k,h,l}$ Let us first consider a particular subgraph S of $S_{k,h,l+1}$, containing $S_{k,h,l}$. It has the same $e_{h,l}$ edges as $S_{k,h,l}$ and in addition, some e edges at level $l+1$, i.e. having $(h-l-1)$ vertices among the x_i's and $(k-h+l+1)$ vertices among the y_j's. Let us compare the densities of S and $S_{k,h,l}$.

$$\begin{aligned} d(S) - d(S_{k,h,l}) &= \frac{e_{h,l} + e}{h + v_{k,h,l} + e(k-h+l+1)} - \frac{e_{h,l}}{h + v_{k,h,l}} \\ &= \frac{e(h + v_{k,h,l} - e_{h,l}(k-h+l+1))}{(h + v_{k,h,l})(h + v_{k,h,l} + e(k-h+l+1))} \ . \end{aligned}$$

From the expressions of $e_{h,l}$ and $v_{k,h,l}$ (formula (5)), it is clear that $h + v_{k,h,l} < e_{h,l}(k-h+l+1)$, iff $l \geq 1$. Hence the density of S is smaller than that of $S_{k,h,l}$. Applying this to $S = S_{k,h,l+1}$, one gets that among all intermediate subgraphs as considered above, that with maximal density is $S_{k,h,1}$.

Take now any non empty subgraph S of $S_{k,h,h}$, with e edges. Let l be the unique integer such that $e_{k,h,l} \leq e < e_{k,h,l+1}$. If S is intermediate between $S_{k,h,l}$ and $S_{k,h,l+1}$, then its density is lower than that of $S_{k,h,1}$ as established above. Otherwise, either one edge of $S_{k,h,l}$ is missing in S, or one edge of S has level $\geq l+2$. In any case, there exists an edge of S with a level strictly higher than the level of an edge missing in S. Let ϵ be an edge with maximal level among those of S and ϵ' be an edge with minimal level among the edges of $S_{k,h,h}$ missing in S. Remove ϵ from S (and the corresponding vertices if they are in no other edge), then add ϵ'. This swapping operation produces a new graph S', such that $d(S') > d(S)$. Indeed, the operation does not change the number of edges, but since ϵ' has a strictly lower level than ϵ, it decreases the number of vertices, hence the density increases. Consider now the algorithm that applies the swapping operation repeatedly while ϵ and ϵ' can be found such that the level of ϵ is strictly larger than that of ϵ'. The density increases at each step, and when the algorithm stops, the output graph is intermediate between $S_{k,h,l}$ and $S_{k,h,l+1}$. This ends the proof of the first part.

The proof of the second part is almost identical. Consider now $S_{k,h,l}$ as a rooted graph with set of roots $R = \{x_1, \ldots, x_h\}$. It satisfies the condition that no edge is included in R, provided $k > h$. Consider again a subgraph S of $S_{k,h,h}$, inbetween $S_{k,h,l}$ and $S_{k,h,l+1}$. Let us compare the density of (R, S) to that of $(R, S_{k,h,l})$.

$$\begin{aligned} d(R,S) - d(R,S_{k,h,l}) &= \frac{e_{h,l} + e}{v_{k,h,l} + e(k-h+l+1)} - \frac{e_{h,l}}{v_{k,h,l}} \\ &= \frac{e(v_{k,h,l} - e_{h,l}(k-h+l+1))}{v_{k,h,l}(v_{k,h,l} + e(k-h+l+1))} \, . \end{aligned}$$

This time the difference is negative for all $l \geq 0$. Therefore, among all intermediate subgraphs, that with highest rooted density is $S_{k,h,0}$, which contains only one edge. The rest of the proof is the same. $\square$

4 The VC and testing dimensions

In this section we prove Theorems 1.2 and 1.3, and deduce from them the asymptotic distributions of the VC and the testing dimensions of a random graph.

Theorem 1.2 states that the threshold of E_h in the model $\mathcal{G}_k(n, p_n)$ is

$$r_n(k,h) = n^{-k+h(h-1)/(h+1)} \, .$$

Proof: The case $h = 1$ is quite elementary and does not actually use Theorem 3.2. Indeed, the VC dimension of a class can only be null if the graph is empty or if it contains a single edge. Therefore the property E_1 is equivalent to "there exist at least two different edges". The number of edges in $\mathcal{G}_k(n, p_n)$ follows a binomial distribution with parameters $\binom{n}{k}$ and p_n. Observe that

$$\binom{n}{k} = \frac{n^k}{k!}(1 + o(1)) \, .$$

The following asymptotics are standard.

- If $p_n \ll n^{-k}$ then the number of edges of $\mathcal{G}_k(n, p_n)$ converges to 0 in distribution.
- If $p_n \gg n^{-k}$ then the number of edges of $\mathcal{G}_k(n, p_n)$ converges to $+\infty$ in distribution.

The result for $h = 1$ follows.

For $h > 1$, as already remarked, the containment of the shattering graph $S_{k,h,h}$ implies E_h. As a consequence of Theorem 2.1 and the first part of Proposition 3.2, the threshold for $Cont(S_{k,h,h})$ is

$$n^{-1/d(S_{k,h,1})} = r_n(k,h) .$$

Thus it only remains to be shown that the same threshold holds for E_h and $Cont(S_{k,h,h})$. Actually, E_h is equivalent to the disjunction of containment properties for all subgraphs G that shatter some set $\{x_1, \ldots, x_h\}$. Any such graph can be deduced from $S_{k,h,h}$ (Definition 3.1), by gluing together vertices among $\{y_1, \ldots, y_{v_{k,h,h}}\}$, belonging to distinct edges in $S_{k,h,h}$: two different edges of G may share some vertices provided they do not belong to $\{x_1, \ldots, x_h\}$. Any such graph G must contain a copy of $S_{k,h,1}$, or of a subgraph denser than $S_{k,h,1}$. Therefore, for any graph G such that $Cont(G)$ implies E_h, $Cont(G)$ has a threshold which is larger or equal to $r_n(k,h)$. Hence if $p_n \ll r_n(k,h)$, no such graph can appear, and E_h is false *w.h.p.* On the other side of the threshold, if $p_n \gg r_n(k,h)$ then $\mathcal{G}_k(n,p_n)$ contains a copy of $S_{k,h,h}$ (by Theorem 2.1) and E_h is true *w.h.p.* $\square$

The VC dimension of the random graph $\mathcal{G}_k(n,p_n)$ is a random variable with values in $\{0, \ldots, k\}$. The property E_h which says that "there exists a shattered set of size h" is equivalent to "the VC dimension is not smaller than h". As a consequence of Theorem 1.2, the asymptotic distribution of the VC dimension of the random graph $\mathcal{G}_k(n,p_n)$ is concentrated on one or two values, according to the values of p_n.

Corollary 4.1 *Let c and α be two positive reals, and assume $p_n = cn^{-\alpha}(1 + o(1))$. Denote by V_k the VC dimension of the random graph $\mathcal{G}_k(n,p_n)$.*

- *If $\alpha > k$ then $V_k = 0$ w.h.p.*

For $h = 1, \ldots, k$,

- *if $\alpha = k - h(h-1)/(h+1)$ then $V_k = h - 1$ or h w.h.p.*
- *if $k - h(h+1)/(h+2) < \alpha < k - h(h-1)/(h+1)$ then $V_k = h$ w.h.p.*
- *If $\alpha < 2k/(k+1)$ then $V_k = k$ w.h.p.*

Notice that the threshold for a maximal VC dimension ($VC = k$) approaches n^{-2} as k increases. So the VC dimension can be maximal even for a relatively sparse graph.

Let us now turn to the testing dimension of the random graph $\mathcal{G}_k(n, p_n)$, which is again a random variable with values in $\{0, \dots, k\}$. Recall that the property A_h (all sets of size h are shattered) is equivalent to "the testing dimension is not smaller than h". Notice that A_k is equivalent to "the graph is complete": this can only occur for $p_n = 1$. Therefore, we shall consider only the case $h < k$. Theorem 1.3 states that the threshold of A_h in the model $\mathcal{G}_k(n, p_n)$ is:

$$s_n(k, h) = (\log n)n^{-k+h} .$$

Proof: Observe that A_h implies that any subset of h vertices in X is contained in a single edge, which is equivalent to $Ext(R, S_{k,h,0})$. By Theorem 2.2, the threshold for that property is $s_n(k, h)$: since $S_{k,h,0}$ has only one edge, the maximal rooted density is $m(R, S_{k,h,0}) = 1/(k - h)$ and the root coefficient is $s(R, S_{k,h,0}) = 1$. Suprisingly enough, part 2 of Theorem 3.2 yields that $s_n(k, h)$ is also the threshold for $Ext(R, S_{k,h,h})$, which implies A_h. Therefore, as soon as p_n is high enough to ensure that any h vertices are in a single edge, then all subsets of size h are shattered and A_h holds *w.h.p.* □

As a consequence of Theorem 1.3, the testing dimension of the random graph $\mathcal{G}_k(n, p_n)$ is asymptotically concentrated on one or two values, according to the values of p_n. For $(\log n)n^{-k+h} \ll p_n \ll (\log n)n^{-k+h+1}$, the testing dimension is h *w.h.p.* For $p_n \gg (\log n)n^{-1}$, it is $k - 1$. Observe also that the threshold $s_n(k, 1) = (\log n)n^{-k+1}$ for A_1 (no vertex is isolated) is also the threshold for connectedness (see [7] for more results on the phase transition of random hypergraphs).

As expected, the threshold of A_h is higher than that of E_h. Curiously, the ratio $s_n(k, h)/r_n(k, h)$ does not depend on k, and not much on h either.

$$\frac{s_n(k, h)}{r_n(k, h)} = (\log n)n^{2-2/(h+1)} .$$

One can also check that

$$s_n(k, h - 2) \ll r_n(k, h) \ll s_n(k, h - 1) .$$

The threshold for A_{h-1} is higher than that of E_h since A_{h-1} implies E_h: if all sets of size $h-1$ are shattered, then at least one set of size h must be shattered. Consider indeed, any set of size h: if all its subsets of size $h-1$ are shattered, then the set itself is shattered as soon as it is contained in one of the edges. On the contrary if p_n is chosen such that $s_n(k, h - 2) \ll p_n \ll r_n(k, h)$, then *w.h.p.* all sets of size $h - 2$ are shattered, but there is no such set of size h.

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