Geometry and Spectral Optimization (GeoSpec)

Heads of the project and participants

Research topics	:	Riemannian Geometry Complex Algebraic Geometry Shape Optimization
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Contents

1 Research Area

The mathematical theory of *complexity* is that of classifying objects or problems based on how difficult they are to apprehend or to solve. It is a well known subject of research in applied mathematics for example in Numerical Analysis or Combinatorics where estimating the number of steps needed to compute a quantity is crucial in the perspective of a computer implementation. In more fundamental mathematics, for example in Geometry or Dynamical Systems, the exact geometry of an object is rarely known with sufficient accuracy: for instance it may be altered with time, as is the case of many mechanical parts, or simply because it is a priori unknown, such as the structure of the Universe. For these reasons, the features of an object are often appraised in terms of associated algebraic or analytic quantities (for instance, the decay rate of the solution of a Partial Differential Equation, etc.), which pave the way to a measure of their complexity.

The purpose of the present project is to develop a synergy between fundamental and applied mathematicians in order to study important problems related to the complexity of two-dimensional objects (such as surfaces) and initiate investigations in higher dimension where the situation is, for most problems, so widely unknown that even stating good questions is very difficult.

In this project we aim at studying the complexity of the geometry of a mathematical object (e.g. a manifold) through three different aspects : its metric (that is, the way distances are measured on the object), its dynamics (i.e. the trajectories followed by particles moving on it) and its spectrum (i.e. its resonance frequencies). More specifically, we intend to study extremal manifolds for invariants describing this complexity in each of the above items.

Historically, the question of studying such extremal manifolds is quite old in theoretical and applied mathematics. The first fundamental results goes back to G. Polya and G. Szegö, who considered the spectrum of a manifold in the particular case where it is simply a subset of the Euclidean space. Later, J. Hersch was able to provide the first result in which the geometry is non trivial, by proving that the first non-trivial eigenvalue of the Laplace-Beltrami operator on a fixed topological class of differentiable manifolds is maximized by the round sphere. In other words the round sphere minimizes the spectral complexity. These investigations have been furthered by several researchers since then, among which M. Gromov, N. Nadirashvili, Y. Sire, etc. Their principle line of research is to find the optimal manifold with respect to its spectral complexity among a certain topological class. For instance it was proved more recently by N. Nadirashvili [23] that the so-called *flat torus* optimizes the first non trivial eigenvalue of the Laplace-Beltrami operator among all orientable surfaces with the same genus and the same measure.

In the next sections, we describe in further details the three main axes of the present project: the study of the metric complexity of objects (Section 3), the understanding of their dynamical complexity (Section 4), and that of their spectral complexity (Section 5).

Before closing this introduction, let us emphasize that the objectives of this project are not purely theoretical. For instance, the use of spectral clustering of geometrical objects in computer science received a particular attention in the last decades (for instance when it comes to shape matching). Many practical questions have been addressed related to the comprehension of the geometry of an object and its spectral data. Whereas progress has been made regarding for instance the parametrization of meshed surface, manifold with a large genus (that is with many holes) are still very challenging to manipulate or cluster. We expect that progresses in the mathematical understanding of the link between the geometry and analytic or dynamical properties may also lead to significant progresses in these fields.

2 Objectives

The general goal of this project is to combine the knowledge of theoretical and applied mathematicians in order to study optimal manifolds with respect to complex criteria, under complex geometrical constraints. The main difficulty to achieve this purpose lies in that the comprehension of extremal manifolds requires fine mathematical parametrization. Related tools have been developed in the areas of topology and dynamical systems but are still beyond the reach of applied mathematicians or computer scientists. This project plans to benefit from the expertise of two teams in Grenoble University in the related fields of theoretical and applied mathematics to produce original and promising studies in this direction: we indeed believe that the researchers of the Fourier Institute, of the LAMA and of the Jean Kuntzmann Laboratory involved in this project have the exact complementary skills which will make it possible to develop new approaches in the numerical approximation of optimal or critical manifolds and the analysis of shape optimization problems under complex geometrical constraints.

We focus our project on three problematics in which both theoretical and applied mathematicians are involved. By this proposal, our team targets progress on understanding deeper the following three different fields of metric structures, dynamical systems and spectral theory.

3 Metric complexity [D.Bucur - G. Besson - É. Oudet - B. Thibert]

Metrics are the relevant mathematical tools to generalize the notion of length, shortest path and curvatures for general surfaces. In the last decades, the theoretical point of view on these classical geometrical notions generated a wide range of applications in many fields like data analysis in high dimension, mesh matching, optimal transportation, etc.

We develop in the following paragraphs the part of our project related to the understanding of abstract metrics. More precisely, we focus here on the identification of specific geometrical objects which help to describe and quantify metric structures.

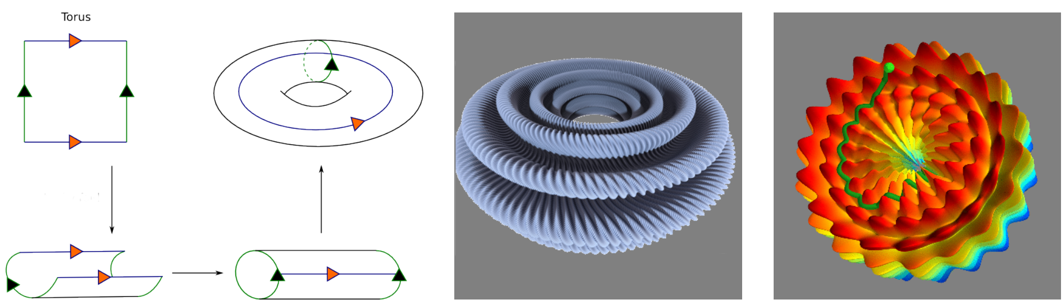


Figure 1: Two approximate isometric embeddings of a flat torus (the left illustration comes from [29] and the central picture has been obtained by from [4]).

3.1 Historical context

There is a long history on isometric embeddings, dating back from the nineteenth century. A map f from a Riemannian surface (M^n, g) into the Euclidean space $\mathbb{E}^q = (\mathbb{R}^q, \langle ., . \rangle)$ is an isometry if it preserves distances. From a practical point of view, the notion of isometric embedding makes it possible to see abstract geometrical object as subspace of the ambient space which both helps its vizualization and the characterization of its complexity.

In some local coordinate system, the identification of an isometric embedding is equivalent to solve a system of non linear partial differential of $s_n = \frac{n(n+1)}{2}$ equations. It was conjectured by Schlaefli [28] in 1873 that any n-dimensional Riemannian manifold can be locally isometrically embedded in $\mathbb{E}^{s_n}$. In the years 1926-1927, Janet and Cartan proved that it is indeed the case if (M^n, g) is an analytic Riemannian manifold. The number s_n is thus called the *Janet dimension.* In this case, every point of M^n has a neighborhood which admits an isometric embedding into $\mathbb{E}^{s_n}$ [18, 9].

In 1954, Nash surprised the mathematical community by breaking down the barrier of the Janet dimension, considering maps with only C^1 regularity [24]. Precisely, he proved that any *strictly short* global embedding $f_0 : (M^n, g) \to \mathbb{E}^q$, can be deformed into a true C^1 global isometric embedding f provided that $q \geq n+2$. Shortly after, the theorem of Nash was extended by Kuiper to the codimension 1 [21].

If we want to increase the regularity of the embedding, one also needs to increase the dimension of the ambient space. Nash showed in 1956 the existence of global C^∞-isometric embedding into $\mathbb{E}^q$ where $q = (3n + 11)n/2$. In the smooth case, the best known general result is due to Gromov and Rokhlin [14] and also Greene [13]. By using a Nash-Moser iteration they proved that local isometric embeddings exist if $q \geq s_n + n$. A better result is known for $n = 2$: any smooth Riemannian surface admits local isometric embeddings into $\mathbb{E}^4$ [26]. For the smooth case, Schlaefli's conjecture is still open even for $n = 2$ (see [16] for a general reference on smooth isometric embeddings, see [31] for an essay in french on the history of isometric immersions).

The flat torus has been the topic of intensive research in computational geometry, the main target of which being to represent numerically the Nash C^1 embedding of the two-dimensional surface in R^3 by constructing it explicitely. In 2012, the so called Hévéa team based in Lyon and Grenoble, developed an analysis of Gromov integration theory which leads to the first vizualizations of the flat torus: see [3] and figure 1. Due to the correlation of the simplicity of the object and the complexity of the required geometrical surface, this work had a huge impact on the mathematical community. This part of the project is mainly dedicated to the investigation of new research directions opened by this work: we would like to pay special attention to the relations between isometric embeddings and spectral properties.

3.2 Main objective: Spectral optimality and isometric embeddings

Hévéa's group investigations have succeeded in the representation of the flat torus and the round sphere (confined in a restrictive space) in $\mathbb{R}^3$. A natural question raised by the interpretation of the results of [23] in this context is the following: can a similar result be obtained by exhibiting directly the optimal surface for the Laplace-Beltrami eigenvalue which is known to be precisely the equilateral flat torus ? Such an approach based on the optimization of a given functional can also be applied in other problems and enable the construction of the embeddings of other surfaces in $\mathbb{R}^3$. This problem is very challenging due to the choice of the numerical representation of the surface, i.e. the choice of parametrization since very singular behavior of optimal surfaces are expected.

The problem of representing the embedded surfaces in $\mathbb{R}^3$ goes beyond the construction of specific examples. In fact the question of finding the optimal $C^{1,\alpha}$ regularity of the Nash embedding is still open. On the other hand the optimal surfaces for the Laplace-Beltrami operator could be expected to achieve the optimal α. We plan in this project to investigate this conjecture both theoretically and numerically: more precisely, we would like to investigate numerically the asymptotic behavior of the Lipschitz constant associated to the Jacobian of the optimal embeddings. Such a refining approach will be possible only through a careful use of smooth spectral representation of the involved approximated isometries.

3.3 Other perspectives

We list below different perspectives that will be developed on this topic by the members of the project.

Optimization in the class of Riemannian metrics. In most of previous theoretical and numerical works, only conformal metrics to a given one are optimized. In this simplified context, the definite positivity is equivalent to impose linear constraints on the unknown factor which reduces dramatically the range of metrics which can be considered. In a general setting, the positivity of the field of matrices impose strongly non linear constraints. We plan to explore the recent results on Monge-Ampere equation and on the discretization of convexity constraint in this context of metric parametrization. Explicit and converging discretization may lead to reliable and fast new numerical schemes in this area based on the use of interior point technics.

Representants of conformal classes. In connection with previous item, we address the question of the numerical identification of specific representants in a family of surfaces. Given a class of surface, for instance defined by a family of parametrization, we ask the question of the algorithmic descritption of some representants of this class which have specific properties (flatness, Hyperbolic, signed curvatures, etc.)

Remeshing in periodic setting. Let S be a surface of genus one and a parametrization ϕ of that manifold by the flat torus. One way to approximate an isometry from the flat torus to that surface is to compose the parametrization ϕ by a diffeomorphism of the torus which leads to a global isometry. This problem may be seen as a remeshing problem for a given map of triangle areas. In order to satisfy asymptoticaly the isometry constraints, some specific remeshing tools have to be developed.

4 Dynamical complexity [C. Dapogny - P. Dehornoy - E. Lanneau - É. Oudet]

4.1 Historical context

Among the class of 3-manifolds, the set of hyperbolic 3-manifolds forms are very rich subclass. One possible way to understand it is by the dynamics of homeomorphisms on surfaces. The link between 2-dimensional dynamics and 3-dimensional topology is given by the so-called *suspension construction*: given a homeomorphism $f : \Sigma_g \to \Sigma_g$ on a two-dimensional surface Σ_g one can form the *suspension* over f namely the 3-manifold $M_f := \Sigma_g \times [0,1]/_{(p,1)\sim(f(p),0)}$.

A deep theorem of Thurston [30] (announced in the 1970s, published in 1988) asserts that when f is "chaotic", the manifold M_f is hyperbolic (*i.e.,* admits a hyperbolic metric). One can then relate the volume $\mathrm{vol}(M_f)$ of M_f to the "disorder" (topological entropy) $\mathrm{ent}(f)$ of f as follows

$$\mathrm{vol}(M_f) \leq 6\pi(g-1) \cdot \mathrm{ent}(f).$$

Several problems of minimization of volumes of hyperbolic 3-manifolds (given a topological class of manifold) are thus related to entropy minimization problems of surface homeomorphisms.

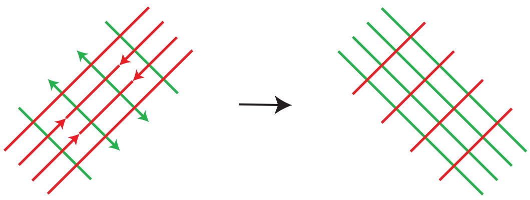
The breakthrough work of Agol [1] says that virtually (up to finite covers) *any* hyperbolic 3-manifold arises in this way! (Agol received the 2016 Breakthrough Prize in Mathematics for this work). Hence homeomorphisms of 2-dimensional real surfaces are fundamental objects that need to be explored!

4.2 Anosov dynamics on surfaces

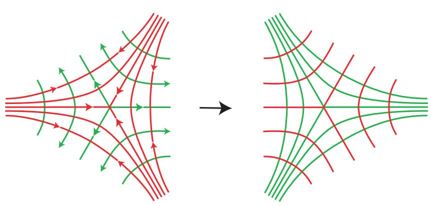
From the dynamical point of view, the simplest chaotic systems are the pseudo-Anosov maps.

Isotopic homeomorphisms have homeomorphic 3-manifolds as suspension. Hence it is enough from the topological point of view to look for a *canonical* representative of every isotopy class. The theorem of Thurston gives a complete answer to this question: every irreducible homeomorphism is isotopic to a *periodic* or to a *pseudo-Anosov* homeomorphism (reducible means in this context that the surface contains invariant curves and the homeomorphism can then be *reduced* to simpler ones). Before explaining what a pseudo-Anosov is, let us underline that this representative is also convenient from the dynamical point of view since pseudo-Anosov maps have the simplest dynamics in their isotopy class: for example every homeomorphism has no fewer periodic points than the pseudo-Anosov to which it is isotopic. Their understanding may reveal behaviors that exist in more complicated systems.

Thurston's classification is transparent on the flat torus, identified with $\mathbb{R}^2/\mathbb{Z}^2$. On this surface every homeomorphism is isotopic to a linear map, given by a matrix in $\mathrm{SL}_2(\mathbb{Z})$. These are the canonical representatives for the torus. Every such matrix has a spectrum that consists of two complex numbers of product 1. If the spectrum lies on the unit circle, then the matrix is actually periodic, as for example $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ which are of order 4 and 6 respectively. If the spectrum consists of two 1's or two -1's, then the matrix is conjugated to $\pm\left(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right)$, which preserves the horizontal direction (reducible case). The last case (and from many points of view the generic and the most interesting one) is when the spectrum consists of two different real numbers $\lambda > 1$ and λ^{-1}, as for example the so-called CAT map given by $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. One then speaks of an *Anosov* homeomorphism. It has two distinct eigendirections E^s and E^u, the first one being contracted and the second one expanded by the same amount λ. The torus then admits two non-singular foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ by lines parallel to these directions. These foliations are invariant under the considered linear map and they are stretched by the factors λ and λ^{-1} respectively. The topological entropy is then $ent(f) = \log(\lambda)$, or $\log(\frac{3+\sqrt{5}}{2})$ for the CAT map.



On general surfaces, the classification goes along the same lines, but one has to extend the notion of Anosov maps. A pseudo-Anosov homeomorphism on a general surface is (in a suitable flat metric) a linear Anosov on the punctured surface. Namely, away from finitely many points, one has two foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ which are invariant under the map and stretched by some factors λ and λ^{-1} respectively. The number λ is called the *expansion factor* and its logarithm coincides with the entropy of the homeomorphism. At the removed points, these foliations have *prongs*:



4.3 Birkhoff sections

As explained above, many 3-manifolds can be presented as the suspension of a pseudo-Anosov homeomorphism of a certain surface. However the surface and the homeomorphism are not unique. For example the 3-dimensional torus $\mathbb{R}^3/\mathbb{Z}^3$ admits infinitely many non homologous embedded 2-dimensional tori (the images under the quotient map of those planes in $\mathbb{R}^3$ with integer coordinates). Cutting the 3-torus along any of these 2-tori shows that the 3-torus is the suspension of some maps of different 2-tori.

For other 3-manifolds there may be even more flexibility as a given 3-manifold M^3 may often be presented as the suspensions of homeomorphisms of surfaces of different genuses! Actually there is one such surface and one associated pseudo-Anosov homeomorphism for every integral homology class in a certain subset F of the second homology group $H_2(M^3;\mathbb{Z})$ called *Thurston's fibered cone* [30]. (In the case of the 3-torus, F is actually the whole group $H_2(\mathbb{R}^3/\mathbb{Z}^3;\mathbb{Z})$).

For every hyperbolic 3-manifold M_f, the homeomorphism $f : \Sigma_g \to \Sigma_g$ is isotopic to a unique pseudo-Anosov representative. Hence the *suspension flow* on M_f, whose orbits are made of vertical segments in the decomposition of M_f as $\Sigma_g \times [0,1]/_{(p,1)\sim(f(p),0)}$ is unique as well.

A key-remark of Thurston is that for a given M, even if there are infinitely many surfaces that decompose M as a suspension, the corresponding flows on M are only of finitely many types (unlike the 2-dimensional example of the torus). In other words M admits finitely many flows $\phi_1, \ldots, \phi_k$ such that all decompositions of M as a suspension are obtained by considering all surfaces transverse to one of $\phi_1, \ldots, \phi_k$.

The set of surfaces transverse to a given flow is rather well understood, and in particular there is a way of finding an optimal surface and homeomorphism: the one that minimizes a quantity called *normalized expansion factor* defined as λ^{2g-2} in the previous notations. Let us call this surface and this homeomorphism canonical. They yield a preferred decomposition of M of the form $S_M \times [0,1]/_{(p,1)\sim(f_M(p),0)}$.

The drawback here is that it is not always easy to determine whether a given homeomorphism of a surface is canonical, and canonical homeomorphisms may be complicated.

The core of our project is to consider an extended notion of section in the hope of simplifying the set of canonical surfaces and homeomorphisms. Namely we want to devote our attention to *Birkhoff sections*: given a flow ϕ_i on a $3-$manifold, one considers a surface with boundary whose interior is transverse to ϕ_i and whose boundary is tangent to it. If the boundary is empty we recover the surfaces and homeomorphisms described above, but in general this gives more flexibility than the sole notion of section without boundary. By an argument of Fried [10], the first-return map on such a section is of pseudo-Anosov type. However in contrary to the case of closed surfaces, it is widely open to understand which homeomorphisms can be obtained, even for the most basic example like the suspension of the CAT map.

The challenging problems on which we plan to work are the following.

1. Describe the *Birkhoff spectrum* of a 3-manifold *i.e.* the set of Birkhoff sections.

2. Does the Birkhoff spectrum always contain a genus 1 surface?

3. How does the Birkhoff spectrum change when taking coverings?

In order to answer these questions (and others), we have two tools in mind. Firstly there are several explicit constructions (following Birkhoff–Fried, Ghys, Brunella, A'Campo–Ishikawa) and it is not yet clear what their range is. A naive approach that may help understanding them is to constructing numerically such surfaces. This is a challenging problem since constructions are not explicit in general. The starting point of our collaboration is to study the geodesic flow on a sphere that is much easier to understand.

Secondly there is an algebraic tool called *Teichmüller polynomial* introduced by Curt McMullen in 2000. This important object controls all the "geometry" and the "dynamics" of the sections. Surprisingly very little is known on this polynomial. Its behavior under removing periodic orbits might be understandable. The difficulty resides in its computation. Again it would help to have a numerical approach to this problem. A master 2 student is already working on this project.

4.4 Other problems

These fundamental questions are naturally related to other important and difficult problems. We mention here the classical Lagrange spectrum. It arises from badly approximable real numbers $\alpha \in \mathbb{R}$ as the set of values of the function

$$L(\alpha) := \limsup_{q,p\to\infty} 1/(q|q\alpha - p|).$$

This function describes the asymptotic depth of penetration of geodesics into the cusp of the modular surface. The structure of L has been studied for more than a century. Much more recently, Hubert-Marchese-Ulcigrai [17] defined the Lagrange spectra for translation surfaces and study their qualitative properties. Pseudo-Anosov elements play an important role in Lagrange spectra, since they provide dense subsets of values. It would be extremely interesting to investigate the relation between Lagrange spectrum and Birkhoff spectrum.

5 Spectral theory: Vector fields, complexity and diffusion [G. Besson - D. Bucur - E. Russ - B. Velichkov]

Consider a compact surface S and a given non-zero vector field V on S (think for example of S as the flat torus and of V as a constant vector field). Suppose that a particle left free on the surface is transported by the vector field V. This system is represented mathematically by the ordinary differential equation $x'(t) = V(x(t))$. Let us consider the following two situations for the trajectory of the particle:

1. Determine the closed periodic trajectories.
2. Determine the trajectories that cannot be confined in any determined zone.

When S is flat and the vector field V is constant the first point is related to the existence of closed geodesics and the related geometric notions of a spectrum discussed in Section 4. The second aspect is related to notions like topological mixing (given an initial distribution of particles, after a finite time one can find a particle in every open set) and mixing (any initial distribution of particles will be spread around to asymptotically form a uniform distribution) also discussed in the previous section. From the PDE point of view we are asking what is the behavior for large t of the solution $\rho(t,x)$ to the transport equation $\partial_t\rho(t,x) + V\cdot\nabla\rho(t,x) = 0$ with a given initial distribution $\rho(0,x) = \rho_0(x)$. In other words we try to quantify the ability of V to concentrate or to spread over S a given distribution of particles. We can accelerate the process by adding a term $\varepsilon\Delta$ which gives rise to the following diffusion process

$$\partial_t\rho(t,x) - \varepsilon\Delta\rho(t,x) + V\cdot\nabla\rho(t,x) = 0.$$

For this equation it is well-known that the behavior of the solution is determined by the first non-trivial eigenvalue $\lambda_1(\varepsilon, V)$, defined as the smallest real number for which the problem

$$-\varepsilon\Delta u + V\cdot\nabla u = \lambda_1(\varepsilon, V)u \quad \text{on} \quad S,$$

has a solution u which is non identically vanishing on S. The eigenvalue $\lambda_1(\varepsilon, V)$ is an indicator of the diffusion properties of V, which can be quantified by the limit $\lim_{\varepsilon\to 0}\lambda_1(\varepsilon, V)$. Thus it is natural to expect that optimizing the first eigenvalue $\lambda_1(\varepsilon, V)$ with respect to V and to S will lead to a vector fields and surfaces with optimal mixing property. On the other hand, the corresponding eigenfunctions indicate the zones in which the diffusion is slow and thus it is natural to expect that theit limit will indicate the (shortest) closed orbits of V.

5.1 Isoperimetric inequalities and spectral optimization. Short review of the problems and the available tools

The isoperimetric inequality. The optimality of a geometric object (open set or manifold) with respect to given criteria can be expressed through an inequality. The classical example is

$$|\Omega| \leq \frac{\text{Per}\,(\Omega)^2}{4\pi}, \quad \text{for all open sets} \quad \Omega \subset \mathbb{R}^2, \tag{5.1}$$

where $|\Omega|$ is the area and Per (Ω) is the perimeter of Ω. The equality in (5.1) is achieved for a disk in $\mathbb{R}^2$ and thus it expresses the fact that among all sets of fixed perimeter the one that has the biggest area is the disk. The above inequality is called *isoperimetric inequality*.

The Faber-Krahn inequality. Several optimization problems involving geometric quantities, in particular spectral quantities (known to control the behaviour of solutions of many PDEs), often originating from physical models, are also referred to as isoperimetric problems ([7]). The first example of spectral optimization goes back to the 19th century and is known as the *the lowest pitch problem* in which we look for a drum of given area that can produce the lowest possible tone. If the drumhead is represented by a domain Ω in the plane, assuming that $\partial\Omega$ (the boundary of Ω) is clamped, the Helmholtz equation shows that the principal frequency at which the membrane can vibrate is the smallest real number $\lambda_1(\Omega)$, called the *principal eigenvalue of the Laplacian on Ω with Dirichlet boundary condition*, for which the following problem:

$$-\Delta u = \lambda_1(\Omega)u \ \text{ in } \ \Omega, \qquad u = 0 \ \text{ on } \ \partial\Omega, \tag{5.2}$$

has a solution $u = u(x, y)$ which is not identically vanishing in Ω. It was conjectured by Lord Rayleigh in 1895 [27] that if the area of Ω is fixed, $\lambda_1(\Omega)$ reaches its infimum when Ω is the disk D. This fact is expressed through the following inequality

$$\lambda_1(\Omega)|\Omega| \geq \lambda_1(D)|D|, \quad \text{for all open sets} \quad \Omega \subset \mathbb{R}^2, \tag{5.3}$$

proved to hold true independently in the 1920's by Faber and Krahn [12, 19, 20].

Rearrangement techniques. The Faber-Krahn inequality (5.3) was first proved by the Schwarz rearrangement technique, which consists in comparing any function u on a domain Ω with a specially constructed test function u^* on the candidate for optimal set (in the case of (5.3), the disk).

A new rearrangement method was recently introduced by Hamel, Nadirashvili and Russ in [15] and applied to the more complex case of an operator with a drift term given by a bounded vector field $V(x) = (V_1(x), \ldots, V_n(x))$ $(n \geq 2)$. Denote by $\lambda_1(\Omega, V)$ the principal eigenvalue of the operator $-\Delta + V \cdot \nabla = -\Delta + \sum_{i=1}^{n} V_i(x) \frac{\partial}{\partial x_i}$ on $\Omega \subset \mathbb{R}^n$ with Dirichlet boundary conditions. Fix $m > 0$ and $\tau \geq 0$. We let Ω and V vary, under the constraints $|\Omega| = m$ and $|V(x)| \leq \tau$ for all $x \in \Omega$. The infimum of $\lambda_1(\Omega, V)$ under these constraints is reached when Ω is a ball B and $V(x) = \tau x/|x|$. Namely, we have the isoperimetric inequality

$$\lambda_1\Big(B, \frac{\tau x}{|x|}\Big) \leq \lambda_1(\Omega, V), \quad \text{for all} \quad \Omega, V \quad \text{such that} \quad |\Omega| = m, \ |V| \leq \tau. \tag{5.4}$$

Direct methods : existence, regularity and computation of optimal sets. In many cases the functionals and the geometric constraints are too complicated and the optimal sets are not known geometric figures. This is the case for example when we consider functionals of the form $\lambda_1 + \cdots + \lambda_k$ or λ_k, for $k \geq 2$ (where the λ_k are the eigenvalues of the Laplacian), that give more precise information on the decay rate of the solutions of the associated diffusion problem. In these cases the existence of optimal sets was proved by Buttazzo-Dal Maso [8], Bucur [5] and Mazzoleni-Pratelli [22] by refined arguments involving techniques from the Potential Theory and Free Boundary Problems. Their regularity ([2, 6, 11]) and geometrical properties, as well as the numerical methods ([25]) are based on the optimality conditions of the form $|\nabla u| = const$ on the free boundary $\partial\Omega$.

5.2 Objectives

In this project we propose to follow the following program (the first three items refer to Section 5):

- For a fixed domain $\Omega \subset \mathbb{R}^n$ or a two dimensional compact flat surface S, establish the relation between the limit $\lambda_1(\varepsilon, V)$, as $\varepsilon \to 0$, and the geometric properties (entropy, mixing, diffusion) of the vector field V and study the limit of the corresponding eigenvalues.

- On a fixed domain or a surface prove the existence of an optimal vector field V_ε and find its properties. Is it measure preserving, parallel, regular? Can we deduce and compute some geometrical properties of the underlying flat surface through the optimal vector field? Prove that the corresponding eigenfunctions converge to the shortest closed geodesics as $\varepsilon \to 0$.

- Find the optimal flat surface and the optimal vector field and study their geometric properties. Establish bounds on the entropy of a generic vector field through the obtained isoperimetric inequality.

- Fix a "box" D (namely a bounded open subset of $\mathbb{R}^n$) and add the constraint $\Omega \subset D$ to the ones in the problem considered in[15]. Is $\inf \lambda_1(\Omega, v)$ reached for some Ω and some v ? If it is the case, what can be said about the minimizing Ω and v ?

6 Request budget and scientific justification

We plan to hire one PhD students (sections 4 and 5) and one post-doctoral student (12 months, section 3). The PhD student will be cosupervised by members of the IF and the LJK like the post-doctoral student. Depending on the background of the student, the PhD thesis will focus on the questions raised in sections 4.3 or 5.2. The post-doctoral student is expected to have skills in both theoretical and applied mathematics to be involved in the research direction raised in section 3.2.

Apart from the supervision, we plan to organize a workshop gathering all members of the team and which could be entitled "Complexity reduction in geometry" and several meetings or one-day workshop with related ANR projects (Comedic, Flat, etc.). In order to organize our research project, we would like to organize at least one day meeting every two months with all members of the project. Our wish would be to synchronize a common invitation every two meetings.

A special semester on "Functional Geometry" will take place at MSRI-Berkley from August 15, 2017 to December 15, 2017. This opportunity will be the occasion of exchanges between people from theoretical geometry and analysts. We strongly believe that a long stay of a member of our group would be the occasion to establish new contacts and identify promising questions where geometry and computer science may fruitfully interact. In the same time it would dramatically increase the visibility of our initiative and be an efficient way to attract very good post-doctoral students.

Other expenses concern participations of members to international conferences of their respective domain. We ask for a total of **287 000 €** (without overhead expenses) for the organization distributed as follows.

Personnel / Staff (Total: 149 300 €)

1. One PhD student for 3 years: 100 965 €.

2. One year of postdoc position: 48 335 €.

Travel & Missions (Total: 47 k€)

1. 27 k€ (9000 € × 3 years for regular visits to foreign colleagues).

2. 20 k€ (Long stay visit of the member and the PhD student).

Invitations (Total: 45 k€) Partitioned as follows:

1. 15 k€ Invitation of young researchers like S. Filip, etc... for 1-2 months each to Grenoble. This is extremely important for establishing ties between young researches for their entire career.

2. 30 k€ Invitation of foreign experts like C. Ulcigrai, Friedl, etc... for 2-3 weeks each to Grenoble. Extremely important for successful development of the project.

Conferences and Schools (Total: 30 k€)
Major international summer school in 2019 at IF & LJK. The past schools gathered about 50 participants correspondingly from all over the world. For this summer school we plan to have at least 100 participants. The requested budget corresponds to approximately 25% of the entire cost. Regular meeting are also planned.

Equipment and operating costs (Total: 15 k€)

1. 10 k€ (3 k€ + 3 k€ + 2 k€ + 2 k€ - personal computers).

2. 4 k€ (memory upgrades, external hard drives, printer and cartridges, software licenses).

3. 1 k€ (books).

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